

THE ELASTIC BEHAVIOUR
OF
THIN CYLINDRICAL SHELLS

by

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A thesis submitted for the degree of
Master of Engineering Science,
in the Faculty of Engineering,
University of Tasmania, Hobart.

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PREFACE

This thesis is concerned with the study of the behaviour of thin cylindrical shells. Besides their many technical uses, cylindrical shells are relatively easy to analyse mathematically, because of their simple geometry; at the same time they exhibit nearly all types of behaviour that one also finds in more complicated shell systems; thus they make a good introduction to the study of shells.

The object is to obtain a basic understanding of the way cylindrical shells carry their loads. Once this understanding is achieved, realistic simplification of the real problem is possible, allowing the setting up of mathematical models. One way of doing this is to observe how a shell deforms through laboratory models.

Cylindrical shells have been extensively investigated mathematically during recent years. The purpose of this thesis is to present the basic elements of shell theory in a simple manner which conforms with modern ideas of structural analysis, and also to study shell behaviour through some simple experimental tests and measurements and numerical calculations. It is believed that, at present, more insight into actual behaviour of structures can be obtained through this approach than through complex mathematical analysis.

The basic mathematical information required for a solution to a shell problem can be conveniently classified into three basic sets of data; the equations of statics, the equations of geometry and the load-deformation relations. These data are developed in Chapter I from basic elementary principles of statics, geometry, and material properties. This treatment is complemented with discussions on the physical significance of each set of equations and its related mathematical problems. Various approximate theories are then derived by making appropriate approximations. This presentation of shell theory is somewhat different from the standard approach and it has some advantages:-

(ii)

(i) Shell theory is put in line with recent ideas of structural analysis; special theories and techniques are no longer being considered in a disconnected manner.

(ii) Nature of any approximation made is clearly understood and can be distinguished as a statical one, a geometrical one, or just the simplification of the element properties under load.

(iii) The relation between the mathematical problems and the practical problems is then seen in a better light.

Although literature on shell theory is abundant, the amount of experimental work reported is small. This is partly due to the difficulties in obtaining measurements on shells. Chapter II of this thesis presents a new technique of measuring slopes on shell surfaces. This is an extension of the Ligtenberg Moire method, from its usual use on flat plates, to cylindrical surfaces. The only previous works in this direction are a paper by Osgeby and a University of Tasmania Honours thesis by Crawford. The technique is further extended and completed in this thesis. Equipment was designed, constructed, and tested by the author. The problem of a concentrated load on a shell edge is solved at the end of the chapter to illustrate the use of the method.

Chapter III is concerned with the Finite Element method. The equivalence of the equations of statics and the principle of minimum potential energy in obtaining the stiffness matrix is demonstrated for the case of a plane rectangular element in plane stress analysis. A cylindrical shell is then presented as an assembly of flat elements. This model has been used previously by Zienkiewicz; however it is modified here to suppress the sixth degree of freedom at each node (rotation in the direction normal to its plane). The programme is also extended to study two useful cases: firstly the problem of prestressed edge beam, and secondly, natural vibrations of shells. Finally some possible extensions of the Finite Element method are discussed; by aligning the elements along lines of certain special characteristics, it is found that the number of elements required is greatly reduced for the same degree of accuracy of the solution obtained. These results are reported in Appendix B for the case of the bending of a plate.

Since design should be the end product of research, the final chapter of the thesis is devoted to design discussions, particularly of concrete shell structures. The design of cylindrical tanks and shell roofs are selected for discussion because of their wide uses. The discussion is partly based on the author's own experience obtained while working with a firm of consulting engineers, and partly based on common practice; possible design applications of the previous studies are indicated where appropriate.

ACKNOWLEDGEMENTS:

The works described in this thesis were carried out during the period May, 1969 to December, 1970, in the Department of Civil and Mechanical Engineering at the University of Tasmania, Hobart, Australia.

I wish to thank all members of the staff of the Department of Civil ^{and} Mechanical Engineering, University of Tasmania, who at sometime or other have been helpful in the fulfilment of this project, in particular I wish to express my deep gratitude for the advice and encouragement received from the following persons:

- Professor A.R. Oliver, Head of Department of Civil and Mechanical Engineering, University of Tasmania.

- Dr. M.S. Gregory, Reader in the Department of Civil and Mechanical Engineering, University of Tasmania - Supervisor of this research.

- Dr. F. Van der Woude, Lecturer in the Dept. of Civil and Mechanical Engineering, University of Tasmania - Joint Supervisor of this research.

- Mr. R.F. Rish, Senior Lecturer in the Dept. of Civil and Mechanical Engineering, University of Tasmania.

- Mr. A.W. Christian of Civil Engineering Laboratory, University of Tasmania.

Thanks is also due to the late Mr. W. Robinson for the care with which he constructed all the testing equipment and models, and Mrs. Y. Barry for preparing this manuscript.

I hereby declare that, except as stated herein, this thesis contains no material which has been accepted for the award of any degree or diploma in any university and that to the best of my belief, this thesis contains no copy or paraphrase of material previously published or written by another person, except where due reference is made in the text of this thesis.

ELASTIC THEORY OF CYLINDRICAL SHELLS

CHAPTER I

1.1 - INTRODUCTION:

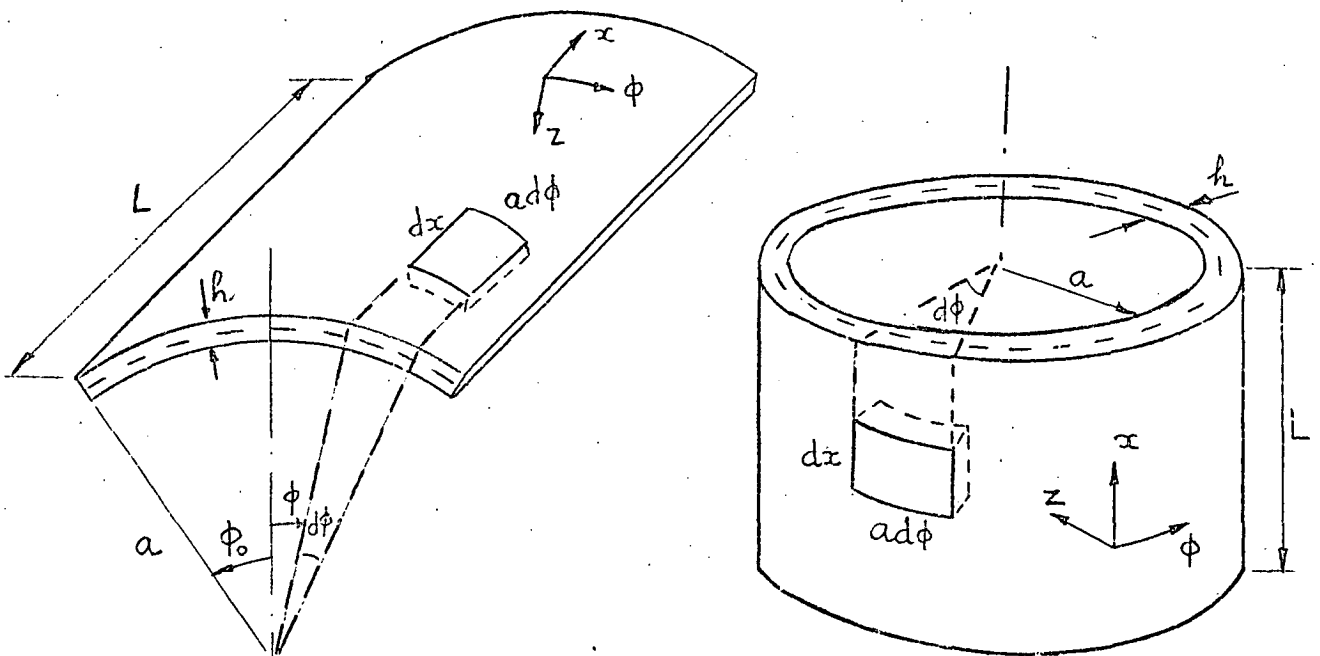
Geometry & Definition:

A cylindrical surface is generated by moving a straight line along a curve while maintaining it parallel to its original direction. The straight line is called the generator and its direction is called longitudinal. The curve is called the directrix and its direction is called transverse.

A cylindrical shell can be complete or incomplete. A shell is complete if the only boundaries that exist are those in the transverse direction.

It will be assumed throughout this thesis that the shell is thin, i.e. its thickness 'h' is small compared with its radius of curvature 'a'. Novozilov¹ suggested $\frac{h}{a} \leq \frac{1}{20}$ as an upper limit for thin shells. Most of the shells used in practice are thin by this criterion.

The surface that bisects the shell thickness is called the middle surface. By specifying this surface and the shell thickness, the shell geometry is completely defined.



This chapter outlines the main elements of the elastic theory of cylindrical shell. It is intended to provide an introduction to the theory of shells and to survey existing literature on the subject. The three basic sets of equations of shells are first introduced. Various approximate methods are then derived from these basic equations by making appropriate assumptions.

1.2 - HISTORY:^{1,2}

The problems of equilibrium and vibration of shells were attempted before the discovery of the general equations of elasticity. Euler attacked the problem of vibration of a bell by dividing it into thin slices and treating each slice as a curved bar - this theory neglected the effect of changing curvature along the axis of the bell. James Bernoulli continued by approximating the bell to two sets of curved bars. Lamé and Clayperon studied the problem of hollow cylinders under internal pressure and symmetrical bending of shells. Lamé's work on curvilinear co-ordinates provided the first simple basis of describing shell geometry. After the general equations of elasticity had been established by Navier, little advance was made on the problem of shells. The special problem of plate received more attention. Poisson and Cauchy proceeded from the general equation of elasticity by expanding the displacements and stresses in terms of power series of the distance from the middle surface. Much controversy arose concerning the convergence of the series and the number of boundary conditions. Kirchhoff introduced the equivalent assumptions of beam theory into plate theory. Under these assumptions the questions of the number of boundary conditions was resolved. Rayleigh considered the vibration problem of shells. He concluded from physical reasoning that the middle surface of a vibrating shell remain unstretched and introduced the theory of inextensional bending of shells. Kirchhoff's method as used for shells was first presented by Aron and Love. Love's works on shells are considered as the basis of modern shell theory. Since Love's time, research on shells has increased enormously and can be broadly divided into two kinds:

- (i) Accurate mathematical theory of shells, which seeks to establish the mathematical foundation of shell theory, to improve and solve the basic sets of equations.
- (ii) Simplified shell theory, which seeks to produce analysis and numerical methods acceptable to design offices, enabling actual shells to be designed and built safely.

1.3-- SHELL BEHAVIOUR:

A shell is a three dimensional structure whose basic strength is given by its form and not by its mass. Transmission of load is done mainly by direct stresses (tension or compression) and in plane shear. Some basic structural properties of shells are best illustrated by comparing them with other structural members. -

A beam resists transverse load by developing bending and transverse shear stresses. A cable can carry the same load by tension alone. An arch is an inverted cable, its main structural action is axial compression. However its bending stiffness is not negligible and substantial bending stresses will arise in most cases. A plate is a two dimensional beam - load is resisted by bending and shear stresses in two directions (for small deflections). A membrane is a two dimensional cable which resists the load through tensile stress only. If a membrane with large deflection is inverted, a shell results - hence the main actions in a shell are direct stresses. The bending stiffness of a shell is, however, finite, therefore bending stresses will inevitably arise. This is the main weakness of shell structures. Shells are usually thin and therefore cannot withstand large bending moments. These bending moments will however exist particularly near the shell boundaries. The determination of stresses is a difficult but important part of shell design calculations.

From experience, it is known that a shell with all the edges free is exceptionally weak, it bends easily without staining the middle surface. A small force will cause large deformations. This phenomenon is called inextensional deformation. A shell can only carry load if its boundaries are adequately stiffened with traverses and edge beams, to prevent any inextensional deformation.

Consider a segment of a cylindrical shell, supported on both ends with traverses, ^{that} carries a uniformly distributed load. This system does not behave as a series of arches because the slender edges cannot take the horizontal thrust of arch action. Its structural action is similar to that of a beam simply supported at both ends. The application of the load on the shell induces longitudinal stresses - compression at the crown and tension at the edges. If the longitudinal span of the shell is long compared with the chord width, the

variation of the fibre stress is the same as that of a normal beam, i.e. linear from crown to edge. As the shell gets shorter, this variation becomes non-linear. The departure from beam theory is basically due to the presence of other structural action especially transverse bending. Thus shell action contrasts sharply with slab action - while a long slab with edge beam carries its load mainly by bending in the shorter direction, a long shell behaves more like a beam.

1.4 Three basic sets of equations of shell -

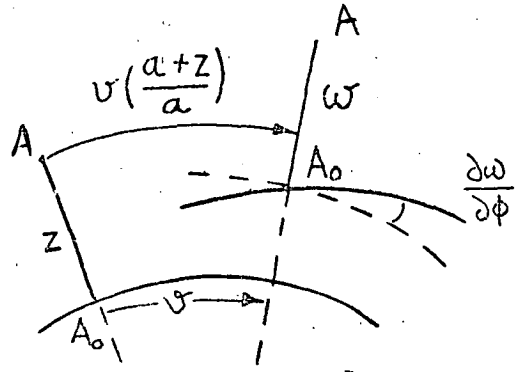
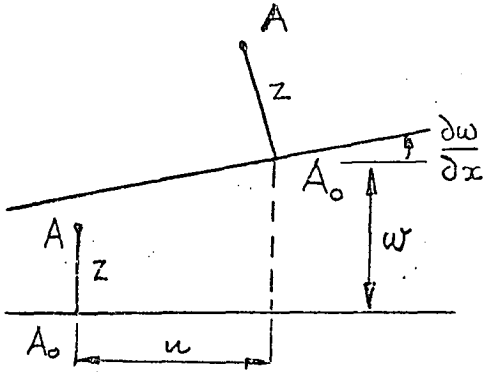
In line with current concepts of structural analysis, all the information required for the solution of shell problems can be distinguished and classified into three basic sets of data. These data are the equations of geometry of deformation, the equations of statical equilibrium, and the load deformation relations or stress-strain relations. This approach has some advantages: special techniques are no longer disconnected, they are merely particular procedures for solving the equations which arise; nature of any approximation made at any level can be clearly understood as a statical one, a geometrical one, or a simplification of the behaviour of the material element under load.

(a) Equations of geometry of deformation.

The undeformed cylindrical shell can be conveniently described by specifying its middle surface and its thickness. The deformation of the shell can, in general, be described by specifying three components of displacement u, v, w at every point within the shell thickness. Subject to the following simplifications, the deformation of the shell can be approximately described by the three components u, v, w of the middle surface only.

- (i) all points lying on one normal to the middle surface before deformation do the same after deformation.
- (ii) the distance of a point from the middle surface may be considered as unaffected by deformation.
- (iii) the normal stress in direction perpendicular to the shell thickness is negligible compared with normal stresses in transverse and longitudinal directions.

Subject to these assumptions, the strains at a point A distance z from the middle surface are expressed in terms of the three displacements u, v, w of the point A_0 projection of A on the middle surface.



Strain in longitudinal direction

$$\epsilon_x = \frac{\partial u}{\partial x} + z \frac{\partial^2 w}{\partial x^2}$$

Strain in transverse direction

$$\epsilon_\phi = \frac{\partial v}{a \partial \phi} + \frac{z}{a(a+z)} \frac{\partial^2 w}{\partial \phi^2} - \frac{w}{a+z}$$

Shear strain

$$= \frac{1}{a+z} \frac{\partial u}{\partial \phi} + \frac{a+z}{a} \frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x \partial \phi} \left(\frac{z}{a} + \frac{z}{a+z} \right)$$

The assumptions made are a generalisation of those usually made in the analysis of beams and plates. They are known as ^{the} Kirchhoff-Love hypothesis. Physically, the first assumption means that the deformations due to transverse shear are neglected. The second and third assumptions are contradictory. If normal stress $\sigma_z = 0$ then due to Poisson's ratio effect, some normal strain ϵ_z must exist and vice versa. Some confusion still exists over this point in shell literature. It will be taken to mean here that whatever happens in the direction normal to the middle surface within the thickness of the shell is insignificant in the overall behaviour of the shell. These assumptions in effect reduce the three dimensional shell problem to that of two dimensions.

Strain - Displacement relations:

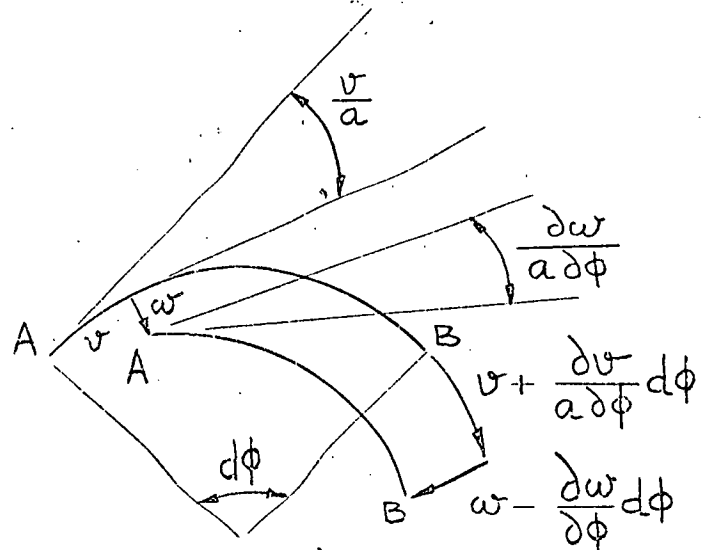
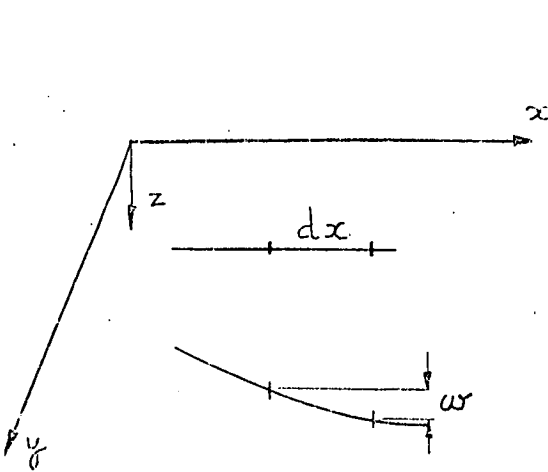
The strains of the middle surface can be expressed in terms of the displacement parameters u, v, w by putting $z=0$ into the above expression.

$$\epsilon_x = \frac{\partial u}{\partial x} \quad \text{--- (1)}$$

$$\epsilon_\phi = \frac{\partial v}{a \partial \phi} - \frac{w}{a} \quad \text{--- (2)}$$

$$\gamma_{x\phi} = \frac{\partial u}{a \partial \phi} + \frac{\partial v}{\partial x} \quad \text{--- (3)}$$

Curvature-displacement relations:



Expressions for curvatures

Rotation of a line element along

Rotation of a tangent

Change in curvature in x-direction

Change of curvature in ϕ -direction

$$\phi_x = \frac{\partial \omega}{\partial x}$$

$$\phi_\phi = \frac{1}{a} \left(v + \frac{\partial v}{\partial \phi} \right)$$

$$\kappa_x = \frac{\partial \phi_x}{\partial x} = \frac{\partial^2 \omega}{\partial x^2}$$

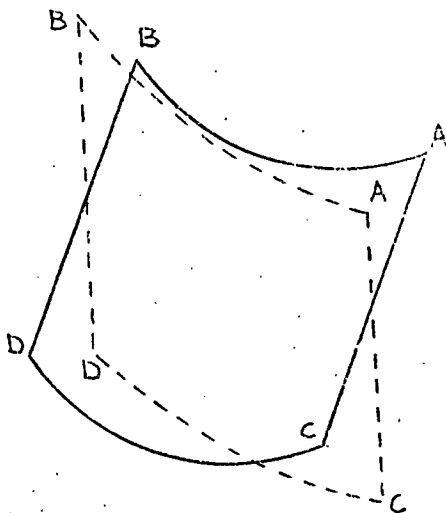
$$\kappa_\phi = \frac{\partial \phi_\phi}{\partial \phi} = \frac{1}{a^2} \left(\frac{\partial v}{\partial \phi} - \frac{\partial^2 \omega}{\partial \phi^2} \right)$$

Bending in ϕ is inextensional, the middle surface does not stretch, thus by the usual curved beam theory $\omega = \frac{\partial v}{\partial \phi}$

Change of curvature in ϕ direction can be rewritten as

$$\kappa_\phi = \frac{1}{a^2} \left(\omega + \frac{\partial^2 \omega}{\partial \phi^2} \right)$$

Expression for twist⁴ - Twist is defined as the rate of change of slope in ϕ -direction as one moves along x-direction or vice versa. Rotation of AB relative to CD is caused by both tangential and radial displacements.



Due to tangential displacement v :

Rotation of AB is $\frac{v}{a}$

Rotation of CD is $\frac{1}{a} \left(v + \frac{\partial v}{\partial x} dx \right)$

Rotation of AB relative to CD is

$$\frac{1}{a} \left(\frac{\partial v}{\partial x} \right) dx$$

Due to radial displacement w

Rotation of AB is $\frac{\partial \omega}{a \partial \phi}$

Rotation of CD is $\frac{1}{a} \cdot \frac{\partial}{\partial \phi} \left(\omega + \frac{\partial \omega}{\partial x} dx \right)$

Rotation of AB vs CD is $\frac{1}{a} \frac{\partial^2 \omega}{\partial \phi \partial x} dx$

The total angular displacement between AB & CD is $\frac{1}{a} \left(\frac{\partial^2 \omega}{\partial \phi \partial x} + \frac{\partial v}{\partial x} \right) dx$

The twist is $K_{x\phi} = \frac{1}{a} \left(\frac{\partial^2 \omega}{\partial x \partial \phi} + \frac{\partial v}{\partial x} \right) \quad (6)$

The equations are basic equations of geometry of deformation. They can be used as such or in another form in which u, v, w parameters are eliminated to give only three equations in terms of strains, curvatures and twist known as the strain compatibility equations:

$$\frac{\partial K_{\phi}}{\partial x} - \frac{\partial K_{x\phi}}{a \partial \phi} = 0$$

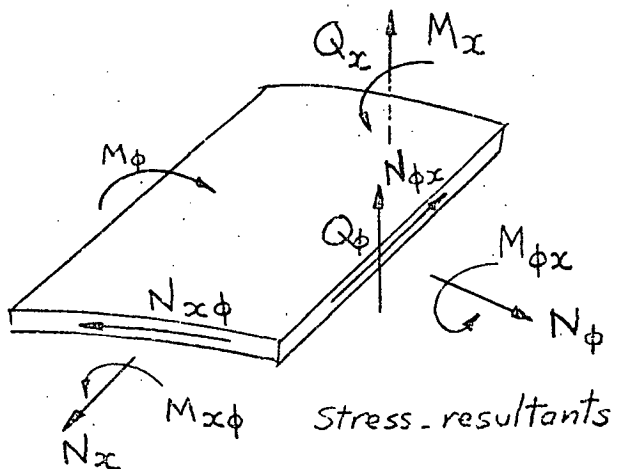
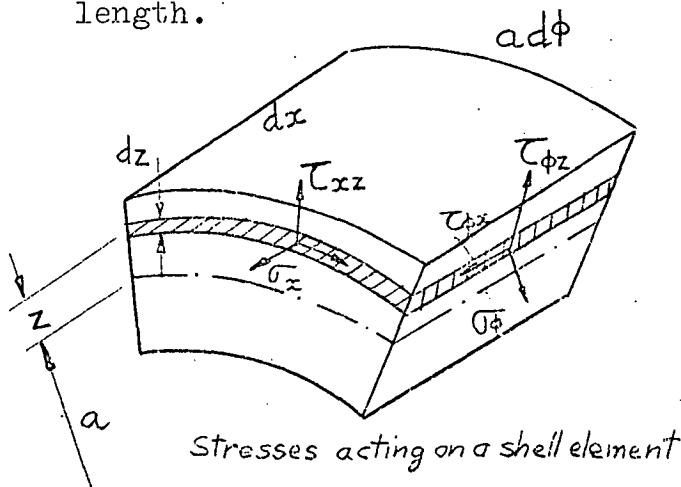
$$\frac{\partial K_x}{a \partial \phi} - \frac{\partial K_{x\phi}}{\partial x} + \frac{1}{a} \left(\frac{\partial \gamma_{x\phi}}{\partial x} - \frac{\partial \epsilon_x}{a \partial \phi} \right) = 0$$

$$\frac{K_x}{a^2} + \frac{1}{a} \left(\frac{\partial^2 \epsilon_x}{a^2 \partial \phi^2} - \frac{\partial^2 \epsilon_{\phi}}{\partial x^2} - \frac{\partial^2 \gamma_{x\phi}}{a \partial x \partial \phi} \right) = 0$$

These equations can be easily verified by substituting expressions for $\epsilon_x, \epsilon_{\phi}, \gamma_{x\phi}, K_x, K_{\phi}, K_{x\phi}$. Physically the equations represent the continuity of the shell surface.

(b) Equations of statical equilibrium

Consider a small shell element, the generalized forces acting on the four cross-sections are called stress resultants. They are usually expressed as the generalized forces per unit length.



The expressions for stress resultants are first derived then the equations of statical equilibrium are obtained in terms of these stress-resultants.

(i) Expressions for stress-resultants -

The stress-resultants are related to the stresses as follows:

$$N_x = \int_{-h/2}^{h/2} \sigma_x \cdot \frac{a+z}{a} dz.$$

$$N_{x\phi} = \int_{-h/2}^{h/2} \tau_{x\phi} \frac{a+z}{a} dz.$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} \frac{a+z}{a} dz.$$

$$M_x = \int_{-h/2}^{h/2} \sigma_z \cdot z \cdot \frac{a+z}{a} dz.$$

$$M_{x\phi} = \int_{-h/2}^{h/2} \tau_{x\phi} \cdot z \cdot \frac{a+z}{a} dz.$$

$$N_\phi = \int_{-h/2}^{h/2} \sigma_\phi dz.$$

$$N_{\phi x} = \int_{-h/2}^{h/2} \tau_{\phi x} dz.$$

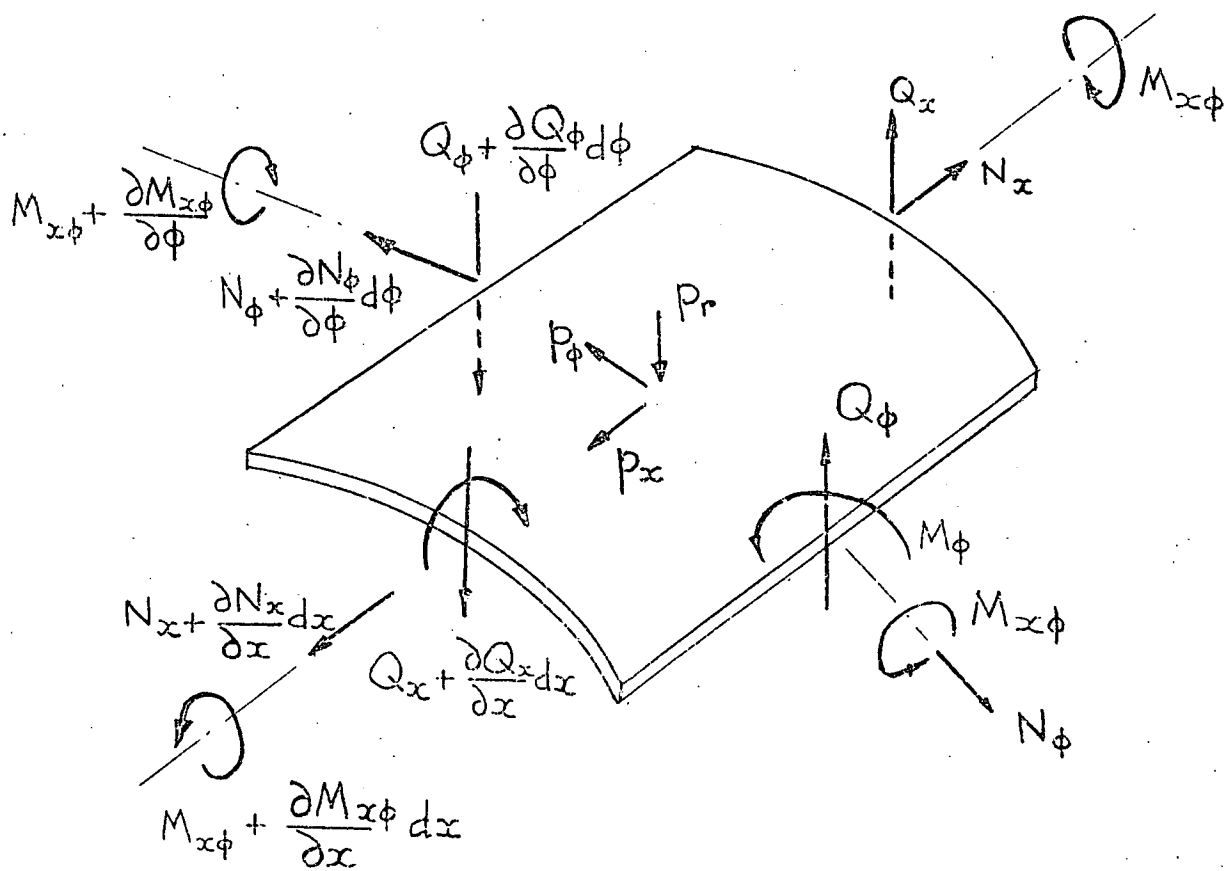
$$Q_\phi = \int_{-h/2}^{h/2} \tau_{\phi z} dz.$$

$$M_\phi = \int_{-h/2}^{h/2} \sigma_\phi \cdot z \cdot dz.$$

$$M_{\phi x} = \int_{-h/2}^{h/2} \tau_{\phi x} \cdot z \cdot dz.$$

The factor $(1 + \frac{z}{a})$ appears in half of the expressions. When the shell is thin $\frac{z}{a} \ll 1$ and thus one might like to neglect it. This simplification gives $N_{x\phi} = N_{\phi x}$ and $M_{x\phi} = M_{\phi x}$. It is very convenient to do this but it has some serious drawbacks which will be discussed later. The factor $(\frac{z}{a})$ represents the effect of the curvature on stress-resultants; thus, neglecting $(\frac{z}{a})$ reduces the expressions to those for flat plates.

(ii) Equations of statical equilibrium



Summation of the forces in x-direction

$$a \frac{\partial N_x}{\partial x} + \frac{\partial N_{x\phi}}{\partial \phi} + p_x \cdot a = 0. \quad (1)$$

Summation of the forces in ϕ -direction

$$\frac{\partial N_\phi}{\partial \phi} - a \cdot \frac{\partial N_{x\phi}}{\partial x} - Q_\phi + p_\phi \cdot a = 0. \quad (2)$$

Summation of the forces in z-direction

$$a \frac{\partial Q_x}{\partial x} + \frac{\partial Q_\phi}{\partial \phi} + N_\phi + p_r \cdot a = 0. \quad (3)$$

Summation of the moments in x-direction

$$- \frac{\partial M_x}{\partial x} - \frac{\partial M_{x\phi}}{\partial \phi} + Q_x \cdot a = 0. \quad (4)$$

Summation of the moments in ϕ -direction

$$- \frac{\partial M_\phi}{\partial \phi} + a \cdot \frac{\partial M_{x\phi}}{\partial x} + Q_\phi \cdot a = 0. \quad (5)$$

Summation of the moments in z-direction

$$a \cdot N_{x\phi} - a \cdot N_{\phi x} + M_{x\phi} = 0. \quad (6)$$

There are ten unknown stress resultants and six equations of statics relating to them. The problem is therefore statically indeterminate. If transverse shear forces Q_x, Q_ϕ are neglected, the first three equations involve only three unknowns, thus constitute a set of statically determinate equations. This is known as the membrane solution, it has an important place in shell theory and will be discussed later.

Q_x, Q_ϕ can also be eliminated using equations (4) & (5), leaving a set of four equations with eight unknowns. In fact this is the only role the transverse shears play in the analysis.

Equation (6) needs some special attention - Generally $M_{x\phi} \neq 0$ thus $N_{x\phi} \neq N_{\phi x}$. It should be recalled that this can be true only if one does not neglect the factor z/a , otherwise this statical condition is violated.

For the case of no surface loading, $N_{x\phi} = N_{\phi x}$, Q_x, Q_ϕ are eliminated using (4) and (5). The equations of statics reduces to three.

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{1}{a} \frac{\partial N_{x\phi}}{\partial \phi} &= 0 \\ \frac{\partial N_\phi}{\partial \phi} + \frac{\partial N_{x\phi}}{\partial x} + \frac{1}{a} \left(\frac{\partial M_{x\phi}}{\partial x} - \frac{\partial M_\phi}{a \partial \phi} \right) &= 0 \\ \frac{N_\phi}{a^2} + \frac{1}{a} \left(\frac{\partial^2 M_\phi}{a \partial \phi^2} + \frac{\partial^2 M_x}{\partial x^2} \right) &= 0 \end{aligned}$$

There is a remarkable resemblance in form of these equations and the equations of strain compatibility with

$$N_x \leftrightarrow K_\phi \quad M_x \leftrightarrow \epsilon_\phi$$

$$N_{x\phi} \longleftrightarrow -K_{x\phi}$$

$$M_{x\phi} \longleftrightarrow \gamma_{x\phi}$$

$$N_{\phi} \longleftrightarrow K_x$$

$$M_{\phi} \longleftrightarrow \epsilon_x$$

This is known as the static-geometric analogy.

In this case the analogy is not complete since one term is still missing in the third equations of statics. The reason is that $N_{x\phi} = N_{\phi x}$ is not admissible. The analogy can be made complete by a redefinition of the shear $N_{x\phi}$ and torsion $M_{x\phi}$.

(c) Load deformation relations -

Some statement must be made about the properties of the material. The material used is considered to be

- (i) Linearly elastic
- (ii) Isotropic and homogenous

Then Hooke's Law is obeyed. In this case since transverse stress σ_z and strains $\epsilon_z, \epsilon_{zx}, \epsilon_{z\phi}$ are neglected, the general three dimensional relations reduce to

$$\sigma_x = \frac{E}{1-\nu^2} (\epsilon_x + \nu \epsilon_{\phi}) ; \quad \sigma_{\phi} = \frac{E}{1-\nu^2} (\epsilon_{\phi} + \nu \epsilon_x) ;$$

$$\tau_{x\phi} = \frac{E}{2(1+\nu)} \gamma_{x\phi} .$$

Hooke's Law is used to express the stress resultants in terms of strains and curvature. The relations are known as generalised stress-strain relations.

Depending on how rigorously one integrates the stress resultant expressions, different results are obtained. The differences are of the order (h/a) and its higher powers, as may be seen in the following tables.

	$z/a \ll 1$ and is neglected during intergration. ⁴	Intergrated expression are expanded in power of h/a fifth and higher order terms are neglected. ³
N_x	$K (\epsilon_x + \nu \epsilon_{\phi})$	$\frac{D}{a^3} K_x$
N_{ϕ}	$K (\epsilon_{\phi} + \nu \epsilon_x)$	$-\frac{D}{a} K_{\phi}$
$N_{x\phi}$	$\frac{K(1-\nu)}{2} \gamma_{x\phi}$	$\frac{D(1-\nu)}{2} (-K_{x\phi} + \gamma_{x\phi})$
$N_{\phi x}$	$\frac{K(1-\nu)}{2} \gamma_{x\phi}$	$\frac{D(1-\nu)}{2} K_{x\phi}$
M_x	$-D (K_x + \nu K_{\phi})$	$-\frac{D}{a} (\epsilon_x + \nu \epsilon_{\phi})$
M_{ϕ}	$-\frac{D}{a} (K_{\phi} + \nu K_x)$	
$M_{x\phi}$	$-\frac{D}{a} (1-\nu) K_{x\phi}$	
$M_{\phi x}$	$-\frac{D}{a} (1-\nu) K_{x\phi}$	$\gamma_{x\phi}/2a$

There is no universal agreement on which expressions are to be used in all cases. The differences between the expressions are mainly on the effects of curvatures on direct stresses, or strains on moments. Koiter⁵ argued, using an energy approach, that Love's first approximation is a consistent one, that in general these differences are quantities of the second order and therefore do not greatly affect the accuracy of the solution. The choice of expression is of considerable value because it allows the selection of the appropriate simplest stress-strain relations for the problem in hand.

1.5 Methods of Solution:

(a) The construction of a set of equations for shells -

Although all the necessary information for the construction of a set of general equations describing shell behaviour have^s been presented, the problem of producing a set of concise, consistent, and adequate equations is still far from being solved - some of the difficulties are summarily discussed.

From Koiter's arguments all shell theories based on Love's assumptions have an inherent error of the order h/a . However, terms of relative order $(h/a)^2$ cannot be generally neglected even though terms of order (h/a) are neglected in the stress strain relation.

Reissner's analysis of the split tube under torsion shows that the relation $N_{x\phi} = N_{\phi x}$ is not acceptable. The equality $N_{x\phi} = N_{\phi x}$ also violates one of the statical equilibrium conditions as mentioned earlier.

The problem of constructing a set of simple, accurate, and consistent equations for cylindrical shell has been a subject of research for many years. This effort is of theoretical rather than practical interest, since the shell equations in their simplest forms are still mathematically very complicated for use.

(b) General methods of solution. Shell problems can be solved by two basic methods common to all structural analysis - in terms of displacements of the middle surface or in terms of stress resultants.

The first method is generally known as the displacement

method. The stress resultants are replaced by their expressions in terms of the strains and hence the displacements of the middle surface. A set of three simultaneous differential equations is obtained involving three unknown displacements u, v, w . These three equations in turn can be reduced to a single differential equation of only one unknown displacement (usually w) by an elimination process. This equation is of 8th order.

The second method is the force method in which the equations of equilibrium are supplemented by three equations of strain compatibilities rewritten in terms of stress resultants. In general, one will have a final set of partial differential equations in terms of six unknown stress resultants. Its solution involves the use of stress functions similar to Airy's stress function in plane stress problems.

In practice, the displacement method is preferred to the force method. The reason for this preference is that the boundary conditions can be easily established in terms of displacements while it is not yet known how to express geometrical boundary conditions in terms of stress functions.

(c) Boundary conditions in displacement method.

The solution of the final eighth order differential equation obtained by ^{the} displacement method consists of two parts - a particular integral and a complementary function. The particular integral involves the loading conditions but requires no boundary conditions. For many shell systems it has been demonstrated that the membrane theory solutions can be used as a particular integral (see Sec. 1.6). The complementary function is the solution of the homogenous equation which involves no loading term. It has eight arbitrary constants which are to be determined by boundary conditions.

The two parts of the solution have physical interpretations. The particular integral (represented by the membrane solution) is the effect of the load on the shell if resistance is provided by membrane stress resultants only. Forces and displacements at the boundary given by this solution will not, in general, be compatible with the known boundary conditions. To remove the incompatibilities, forces and displacements must be applied to the edges (edge effects), these are provided by the determination of the eight constants of the complementary function.

1.6. APPROXIMATE METHODS IN CYLINDRICAL SHELL THEORY:

Introduction -

Shell equations as presented above are lengthy and difficult to solve. Various attempts have been made to reduce the equations to a form more suitable for design purposes. More drastic approximations must be made.

The art of approximation lies in the selection of terms to be neglected. Terms which have no important effects, and introduce unwarranted difficulties in the analysis, should be left out. These terms may vary from problem to problem and therefore it is important that the region in which a certain solution is applicable must be clearly defined.

The engineer is entitled to use all information available to him either by mathematical reasoning, physical intuition, or experimental measurements to make a judgment about the relative importance of various terms and the quickest way to obtain a solution. Most of the calculations nowadays can be done on a computer; reasonably accurate solutions are possible in relatively short time. However, the answer must be checked over by the engineer, thus there is still a need for quick approximate manual methods for checking purposes. These methods are most appropriately based on experimental measurements. In this section, some of the well known approximate methods are examined. Poisson's ratio will be neglected for simplicity but it can be reintroduced without any undue complication.

Membrane Theory -

Assumption - The shell behaviour is approximated to that of a membrane, i.e. there are no bending stress and transverse shear forces.

Mathematically $M_x, M_\phi, M_{x\phi}, Q_x, Q_\phi$ are neglected.

Analysis -

(i) Equation of statics - When the above terms are neglected, there remains only three equations of statics with three unknown stress resultants, $N_x, N_\phi, N_{x\phi}$. The problem is thus statically determinate

$$N_\phi = a p_r \quad \text{-----} \quad (1)$$

$$\frac{\partial N_x}{\partial x} + \frac{1}{a} \frac{\partial N_{x\phi}}{\partial \phi} = - p_x \quad \text{-----} \quad (2)$$

$$\frac{1}{a} \frac{\partial N_\phi}{\partial \phi} + \frac{\partial N_{x\phi}}{\partial x} = - p_\phi \quad \text{-----} \quad (3)$$

(ii) Expression for stress resultants - Stress resultants are found by solving the equation of statics without using the equation of geometry because the problem is statically determinate - N_ϕ is given directly by (1)

$$N_{x\phi} = - \int \left(p_\phi + \frac{1}{a} \frac{\partial N_\phi}{\partial \phi} \right) dx + f_1(\phi)$$

$$N_x = - \int \left(p_x + \frac{1}{a} \frac{\partial N_{x\phi}}{\partial \phi} \right) dx + f_2(\phi)$$

For common loading cases where

$p_x = 0$, i.e. no longitudinal loading, and p_ϕ & p_r are independent of x , then

$$N_{x\phi} = -x F(\phi) + f_1(\phi) \quad , \quad \text{where}$$

$$F(\phi) = p_\phi + \frac{1}{a} \frac{\partial N_\phi}{\partial \phi}$$

$$N_x = \frac{x^2}{2a} \frac{dF(\phi)}{d\phi} - \frac{x}{a} \frac{df_1(\phi)}{d\phi} + f_2(\phi)$$

For a complete shell supported on diaphragms, the distribution of longitudinal stress N_x and in plane shear $N_{x\phi}$ is similar to the distribution of moment and shear for beam. These cases have been considered very clearly by Flugge.

(iii) Deformation due to membrane forces - Using the equations of geometry and stress strain relations, the displacements due to membrane theory are determined.

$$u = \frac{1}{Eh} \int N_x dx + f_3(\phi)$$

$$v = -\frac{1}{a} \int \frac{\partial u}{\partial \phi} dx + \frac{2}{Eh} \int N_{x\phi} dx + f_4(\phi)$$

$$w = \frac{\partial v}{\partial \phi} - \frac{aN_\phi}{Eh}$$

(iv) Note on membrane theory -

(a) Region of application - For membrane theory to be applicable, the shell must be complete. In such a shell, the shell shape conforms to an internal network of forces which follows the curved shape of the membrane. An

example of a pure membrane state of stress is a pipe subjected to fluid pressure. If the shell is not complete, e.g. shell roofs, or the load is not uniformly distributed, e.g. ring load on pipe, membrane stresses alone are not sufficient to maintain equilibrium. Bending stresses are set up. In general membrane theory gives a good insight into the structural behaviour of the shell, its stress productions are good in those regions away from boundaries, concentrated load, cut-outs, stiffening ribs.

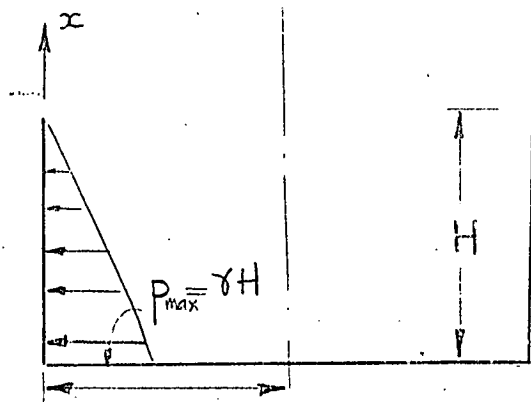
(b) Structural actions of cylindrical shell according to membrane theory - A close investigation of the stress resultant expressions shows that N_ϕ is independent of the boundary conditions and that it is the same for all cross sections. N_ϕ is dependent on radial loading p_r only. Thus, if $p_r = 0$, then $N_\phi = 0$ at the edge. This can be achieved by using a shell with a vertical tangent. However $N_{x\phi}$ does not vanish, therefore it is still necessary to provide an edge member to transmit the shear. It is also possible to choose a shell shape so that the load is transmitted wholly in the transverse direction. If the cross-section is a funicular curve, the shell becomes a series of arches and beam action completely disappears, e.g. cycloidal shell under uniform vertical load.

(c) Use of membrane theory - In preliminary design, membrane theory is used to estimate the overall shell dimensions and thickness. Various cylindrical profiles can be tried, membrane stresses are calculated to find out the most suitable shape, e.g. circular, parabolic . . .

It can also be used as a first approximate solution upon which corrections will be made to determine the true stress distributions. For the general solution of the differential equation governing bending behaviour of shell, membrane theory can be used as a particular integral without any great loss of precision.

(v) Example -

(i) Circular tank under hydrostatic pressure



If both ends of tanks are free to slide and rotate then membrane theory is applicable.

$$p_r = -\gamma(H-x)$$

Where γ = specific weight of liquid

$$N_\phi = -a p_r = a\gamma(H-x)$$

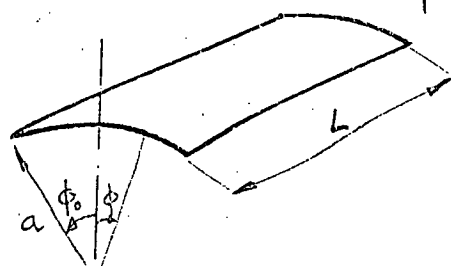
Displacement of tank according to membrane theory

$$\epsilon_\phi = \frac{N_\phi}{Eh} = -\frac{w}{a} \Rightarrow$$

$$\text{Radial deflection } w = \frac{\gamma a^2}{Eh} (H-x)$$

$$\text{Longitudinal slope } \frac{\partial w}{\partial x} = \frac{\gamma a^2}{Eh}$$

(ii) Barrèl Vault - Circular barrel vault under uniform vertical load p .



$$p_r = -p \cos \phi ; p_\phi = p \sin \phi ; p_x = 0 ;$$

$$N_\phi = -p a \cos \phi ; N_{x\phi} = -2 p x \sin \phi ;$$

$$N_x = \frac{p}{4a} (L^2 - 4x^2) \cos \phi$$

Displacement of barrel vault due to Fourier series loading:

$$p = \frac{4}{\pi} p_d \sin \frac{n\pi x}{L}$$

$$\text{then } u = \frac{2p}{Eha} \cdot \frac{1}{k^3} \cdot \cos \phi \cdot \cos kx$$

$$v = -\frac{2p}{Eh} \left(\frac{1}{a^2 k^4} + \frac{2}{k^3} \right) \sin \phi \cdot \sin kx$$

$$w = \frac{2p}{Eha^3} \frac{1}{k^4} \left(1 + 2a^2 k^2 + \frac{a^4 k^4}{2} \right) \cos \phi \cdot \sin kx$$

$$\text{where } p = \frac{4}{\pi} p_d ; k = \frac{n\pi}{L}$$

Vertical deflection

$$\Delta_v = -v \sin \phi + w \cos \phi = \frac{2p}{Eha^3} \frac{1}{k^4} \left[1 + 2a^2 k^2 + \frac{a^4 k^4}{2} \cos^2 \phi \right] \sin kx$$

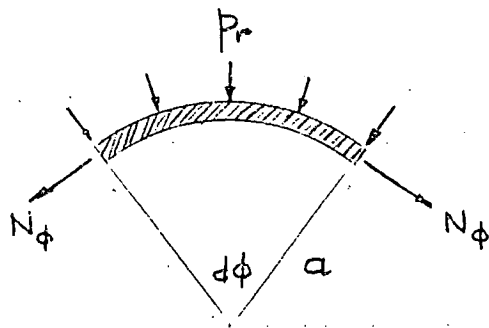
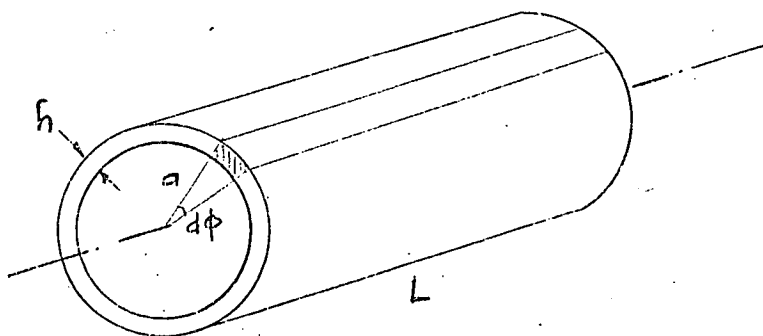
Horizontal deflection

$$\Delta_H = v \cos \phi + w \sin \phi = \frac{pa^2}{Eh} \sin \phi \cdot \cos \phi \cdot \sin kx$$

Beam on Elastic Foundations Analogy -

(a) Assumptions

This method was established for axisymmetric loading for complete shell but it can be used in a wider context which will be discussed later. Axisymmetric loading allows the following simplifications:



- (i) No tangential and longitudinal load $p_x = p_\phi = 0$.
- (ii) No longitudinal and in plane shear stresses $N_x = N_{x\phi} = 0$.
- (iii) No transverse & twisting moment $M_x = M_{x\phi} = 0$.

(b) Analysis

- (i) Equations of statics - when the above terms are neglected, the equations of statics reduce to:

$$Q_x - a \frac{\partial M_x}{\partial x} = 0$$

$$\frac{\partial Q_x}{\partial x} - N_\phi = -p_r \cdot a$$

- (ii) Equation of geometry

$$\epsilon_\phi = \frac{w}{a}$$

$$\kappa_x = \frac{\partial^2 w}{\partial x^2}$$

- (iii) Stress strain relations

$$N_\phi = K \epsilon_\phi$$

$$M_x = -D \kappa_x$$

- (iv) Differential equation. Equations of statics are reduced to one, substitution of equations of geometry and stress strain relations give the following differential equation:

$$D \frac{d^4 w}{dx^4} + K \frac{w}{a^2} = -p_r$$

where $D = \frac{E h^3}{12}$; $K = E h$.

Equation is rewritten as

$$\frac{d^4 w}{dx^4} + 4\lambda^4 w = \frac{12 p_r}{E h^3} \quad \text{where} \quad 4\lambda^4 = \frac{12}{h^2 a^2}$$

Solution to the above equation is

$$w = p(x) + e^{-\lambda x} (C_1 \cos \lambda x - C_2 \sin \lambda x) + e^{\lambda x} (C_3 \cos \lambda x - C_4 \sin \lambda x)$$

where $p(x)$ is a particular integral : $p(x) = \frac{a^2}{E h} p_r$.

This particular solution represents the state of membrane stresses in which there is no bending.

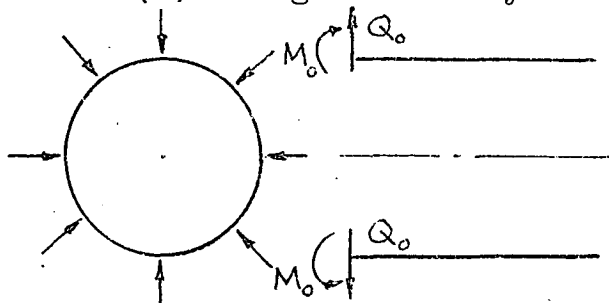
The general solution has four constants which are to be determined by boundary consideration. It shows the influence of the edges which may be interpreted as two waves, each emanating from an edge. These waves dampen quickly. The nature of the damping is characterised by λ . In practice the edges do not influence each other since each edge has one wave which is dampened out quickly, therefore $C_3 = C_4 = 0$.

(c) Note on beam on elastic foundation analogy

By virtue of axial symmetry, the two dimensional shell problem has been further reduced to one dimensional. It is the simplest bending solution for shells. It also exhibits the general characteristics of bending of shell (excluding inextensional bending) i.e. any bending moment is quickly dampened out and therefore bending state in shell tends to be a local problem confined largely to a small region near the disturbance. Haas⁸ and Bouma¹¹ have shown that this characteristic is common to all shells of positive curvature. They suggested a general approximate method to evaluate the edge effects of this kind by using the concept of effective widths and characteristic lengths with similar governing differential equation.

(d) Example

(i) Ring load on cylinder - As shown, the above solution is



$$w = e^{-\lambda x} (C_1 \cos \lambda x + C_2 \sin \lambda x)$$

$$(M_x)_{x=0} = -D \frac{d^2 w}{dx^2} = M_0$$

$$(Q_x)_{x=0} = \frac{dM_x}{dx} = -D \frac{d^3 w}{dx^3} = Q_0$$

$$C_1 = \frac{1}{2\lambda^3 D} (Q_0 + \lambda M_0) ; C_2 = \frac{M_0}{2\lambda^2 D}$$

$$w = \frac{e^{-\lambda x}}{2\lambda^3 D} [\lambda M_0 (\sin \lambda x - \cos \lambda x) - Q_0 \cos \lambda x]$$

$$\left(\frac{\partial w}{\partial x} \right)_{x=0} = \frac{1}{2\lambda^2 D} (2\lambda M_0 + Q_0)$$

$$(w)_{x=0} = -\frac{1}{2\lambda^3 D} (\lambda M_0 + Q_0)$$

For ring load P

$$Q_0 = -\frac{P}{2}$$

At the point of application, the longitudinal slope is zero.

$$\left(\frac{\partial \omega}{\partial x}\right)_{x=0} = 0 \Rightarrow 2\lambda M_0 - \frac{P}{2} = 0$$

$$M_0 = \frac{P}{4\lambda}$$

It is interesting to note that this method gives the moment at the point of application as $M_x = \frac{P}{4\lambda}$ and also the assumptions imply that direct longitudinal stress at this point is negligible $N_x = 0$. In section (1.4), the stress strain relation, for N_x is shown to be

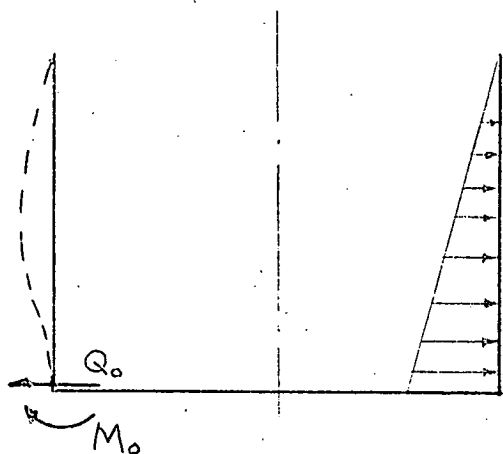
$$N_x = K \epsilon_x - \frac{D}{a^2} \kappa_x$$

$$N_x = K \epsilon_x - \frac{M_x}{a^2} \neq 0$$

This example does not invalidate the result of the analysis, it only shows the approximate nature of the solution.

(Numerically $\frac{M_x}{a^2}$ is a small value). The same sort of discrepancy might be detected in many similar problems.

(ii) Circular tank under hydrostatic pressure



In this case $p_r = -\gamma(H-x)$

Thus the particular integral becomes

$$p_x = -\frac{\gamma(H-x)}{Eh} a^2$$

same as that given by membrane theory.

The two constants of the complementary function be obtained by boundary condition.

In a fixed end tank, the boundary conditions are -

no radial displacement
no rotation } at lower edge

Alternatively one can take the membrane solution as a first guess; the errors in the membrane solution are then the displacement at the lower edge. To correct this one needs to apply a shear Q_0 & a moment M_0 at the end of tanks.

$$(\omega)_{x=0} = -\frac{1}{2\lambda^2 D} (\lambda M_0 + Q_0) - \frac{\gamma a^2 h}{Eh} = 0$$

$$\left(\frac{\partial \omega}{\partial x}\right)_{x=0} = \frac{1}{2\lambda^2 D} (2\lambda M_0 + Q_0) + \frac{\gamma a^2}{Eh} = 0$$

Beam Arch Theory -

Assumptions - The beam arch theory approximates the behaviour of a shell to that of a beam in the longitudinal direction and to that of an arch in the transverse direction. This implies the following factors may be taken as negligible.

- (i) Deformation of the cross-section
- (ii) Longitudinal bending moments M_x , torsional moments $M_{x\phi}$ and radial shear forces Q_x
- (iii) Strain from in plane shearing forces $\gamma_{x\phi}$

Analysis - The analysis consists of two parts

(i) Beam Analysis - The shell is regarded as a beam of curved section. Beam formulae are used to estimate the longitudinal stress N_x and in plane shear

$$N_x = \frac{M \bar{z}}{I} \quad N_{x\phi} = \frac{VQ}{Ib}$$

For a symmetrical section

$$\bar{z} = \frac{a \sin \phi_0}{\phi_0} \quad I = a^3 h \left[\phi_0 + \sin \phi_0 \left(\cos \phi_0 - \frac{2 \sin \phi_0}{\phi_0} \right) \right]$$

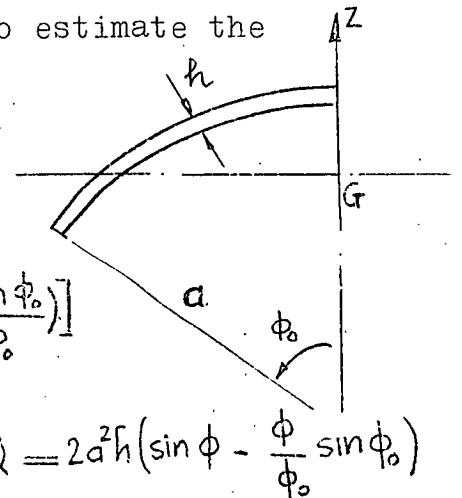
(ii) Arch Analysis

For a slice of shell dx , the vertical loads p are held in equilibrium by the vertical component of the shear

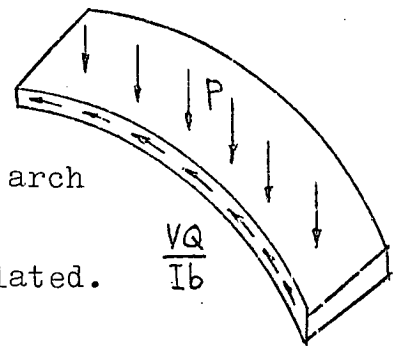
$$\frac{dVQ}{Ib} \sin \phi = \omega \frac{Q}{Ib} \sin \phi$$

The moment M_ϕ at any point is found as the sum of the moments of the loading and of the vertical shear components by usual arch analysis.

When M_ϕ is known, Q_ϕ and N_ϕ can be calculated.



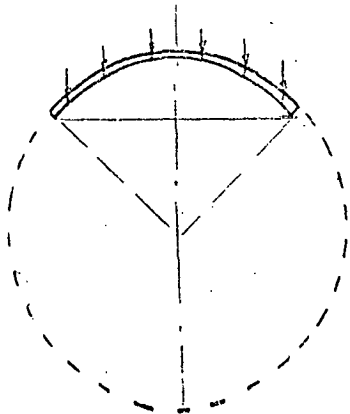
$$Q = 2a^2 h \left(\sin \phi - \frac{\phi}{\phi_0} \sin \phi_0 \right)$$



Note on beam arch theory

(i) Discussion on the assumptions of the theory - The first assumption is that "plane section remains plane" which is the heart of beam theory. The stress resultants neglected in the second assumption are usually small, however no statement can be made on its importance in the derivation of the other stress resultants such as transverse moments and stresses $N_x, N_{x\phi}$. The third assumption is usually made in flexural analysis. In general, these assumptions violate the basic shell equations but if they are approximately true, then beam theory can be used with reasonable confidence.

(ii) Relation between beam theory and membrane theory⁷



$$Q = 2ha^2 \sin \phi$$

$$I = \pi a^3 h$$

$$\sigma_x = \frac{N_x}{h} = - \frac{M y}{I}$$

$$\tau_{x\phi} = \frac{N_{x\phi}}{h} = - \frac{VQ}{2Ih}$$

Consider the barrel vault section as part of a complete shell subjected to the same loading intensity. Beam analysis of this hollow beam gives:

$$M = p \cdot 2\pi \cdot a \cdot \frac{L^2 - 4x^2}{8}$$

$$y = a \cos \phi$$

$$V = p \cdot 2\pi \cdot a \cdot x$$

$$N_x = \frac{p \cdot 2\pi \cdot a (L^2 - 4x^2) a \cos \phi \cdot h}{8 \pi a^3 h}$$

$$= \frac{p}{4a} \cos \phi (L^2 - 4x^2)$$

$$N_{x\phi} = \frac{p \cdot 2ha^2 \sin \phi \cdot 2\pi a \cdot x}{2\pi a^3 h}$$

$$= 2p \sin \phi \cdot x$$

These results are identical with those given by membrane theory.

(iii) Application of Beam Arch Theory

As the assumptions imply, beam theory can only be used when the shell is fairly long. Chinn⁹ stated that the beam method can be used for any shell provided that the assumptions made are justifiable. He studied the case of ratio of length to radius $\frac{L}{a} = 1.1$ and $\frac{L}{a} = 2$ and found satisfactory good results in both. Parme and Corner¹⁰ in a subsequent discussion on the same paper suggested that limit. They also provided tables for symmetrically loaded shell which gives values directly.

Beam arch theory is applicable to any kind of cross-section while other bending theory methods tend to be restricted to the circular shells. It can be modified to be used for shells with non uniform thickness or shells reinforced with ribs and edge beam. Chronowicz¹⁷ claims that line load on shell can also be treated this way. Ramaswamy¹² analyses north light shell with beam arch theory.

Although the beam calculation is simple and provides quick and reasonably accurate information on longitudinal stress, the arch calculation is usually lengthy and tedious. Calculations must be performed at a very high degree of accuracy, as most of the values are small differences of large quantities.

Bending Theory of Cylindrical Shell -

Introduction - The most exact theory for the analysis of bending of cylindrical shell is that proposed by Flugge. Analyses based on Flugge's equations are long and complicated. Various attempts have been made to simplify the calculations such as the works of Aas-Jakobsen, Finsterwalder, Donnell, Schorer.... One common feature of all these theories is that the loading is expressed in terms of Fourier series. With this loading, the stress resultants and deformations will be in trigonometric form and the boundary conditions at the simply supported ends are automatically satisfied. In this section, the two most well known shell bending equations are presented, the Schorer's equation and the Donnell's equation.

(a) The Schorer Theory:

1. Assumptions

(i) Longitudinal moment M_x and shear Q_x are small. Torsional moment $M_{x\phi}$ is also neglected.

(ii) Tangential strain ϵ_ϕ and shear strain $\gamma_{x\phi}$ are small compared with longitudinal strain ϵ_x

This allows a simplification in the equation of geometry.

$$\omega = \frac{\partial v}{\partial \phi} \quad ; \quad \frac{\partial v}{\partial x} = -\frac{1}{a} \frac{\partial u}{\partial \phi}$$

2. Analysis

Subject to the above simplification, the three basic sets of equations reduce to

(i) Equations of statical equilibrium

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{1}{a} \frac{\partial N_{x\phi}}{\partial \phi} &= 0 \\ \frac{\partial N_\phi}{\partial \phi} + a \frac{\partial N_{x\phi}}{\partial x} &= 0 \\ N_\phi + \frac{\partial Q_\phi}{\partial \phi} &= 0 \\ \frac{1}{a} \frac{\partial M_\phi}{\partial \phi} - Q_\phi &= 0 \end{aligned}$$

(ii) Equation of geometry

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} & K_\phi &= \frac{\partial^2 \omega}{\partial \phi^2} \\ \omega &= \frac{\partial v}{\partial \phi} & \frac{\partial v}{\partial x} &= -\frac{1}{a} \frac{\partial u}{\partial \phi} \end{aligned}$$

(iii) Stress strain relation

$$\begin{aligned} M_\phi &= -\frac{D}{a^2} K_\phi \\ N_x &= E h \cdot \epsilon_x \end{aligned}$$

(iv) Schorer equation

Eliminate N_ϕ , Q_ϕ , $N_{x\phi}$ from the equation of statics then use stress strain relations and equations of geometry to obtain Schorer equation. It can be written in terms of the radial displacement w or longitudinal displacement u .

In terms of w

$$\frac{\partial^8 w}{\partial \phi^8} + \frac{a^4}{k} \frac{\partial^4 w}{\partial x^4} = 0$$

where

$$k = \frac{h^2}{12a^2}$$

In terms of u

$$\frac{\partial^8 u}{\partial \phi^8} + \frac{a^4}{k} \frac{\partial^4 u}{\partial x^4} = 0$$

(v) Solution -

The solution is of the form

$$w = A e^{m\phi} \cos \frac{\lambda_n x}{a}$$

The characteristic equation is

$$m^8 + \frac{\lambda_n^4}{k} = 0$$

This equation has 8 roots

$$\begin{aligned} m_1 &= \alpha_1 + i\beta_1 = -m_5 \\ m_2 &= \alpha_1 - i\beta_1 = -m_6 \\ m_3 &= \alpha_2 + i\beta_2 = -m_7 \\ m_4 &= \alpha_2 - i\beta_2 = -m_8 \end{aligned}$$

Expression for w

$$w = [e^{-\alpha_1 \phi} (A_n \cos \beta_1 \phi + B_n \sin \beta_1 \phi) + e^{-\alpha_2 \phi} (C_n \cos \beta_2 \phi + D_n \sin \beta_2 \phi)] \cos \frac{\lambda_n x}{a}$$

Expression for other stress resultants and

displacement can be derived from expression for w .

3. Remark on Schorer's equation

(i) Range of validity - Schorer's equation is in fact a radical condensation of the Finsterwalder's equation.¹² Schorer himself limited the analysis to shall having a span to radius ratio $\frac{L}{a} \geq \pi$: the shell is fairly long. This limitation is also implicit in the assumptions that tangential strains and shear strains are small. Tottenham¹⁴ stated that Schorer's equation is applicable for most "practical" shells with insignificant error except for the class of short shell where radius and chord widths are much greater than the span. Thus Schorer's equation has wider range of application than Schorer' himself anticipated.

(ii) Advantages of Schorer's equation - For the class of shell where the equation is applicable, it offers the simplest possible analysis of bending of shells. When written in terms of radial displacement w , the equation is particularly suitable for the analysis of long shell with reinforced edge beam or prestressed edge beam. When written in terms of longitudinal displacement u , it is suitable for use in problems where there are irregular change in longitudinal strain, e.g. tank on elastic foundation, irregular support etc.

(b) ¹³ The Donnell Theory -

1. Assumption - The Donnell theory approximates the structural action of a shell to that of a disk (in plane action) a plate (bending action) and a membrane.¹² (membrane action.) This theory takes into account M_x , $M_{x\phi}$, Q_x which are neglected in Schorer's theory. However it neglects $\frac{\partial N_\phi}{\partial \phi}$ when it occurs together with $\frac{\partial N_{x\phi}}{\partial x}$.

2. Analysis

(i) Equations of statical equilibrium

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{1}{a} \frac{\partial N_{x\phi}}{\partial \phi} &= 0 \\ \frac{\partial N_\phi}{\partial \phi} + a \frac{\partial N_{x\phi}}{\partial x} &= 0 \\ a \frac{\partial Q_x}{\partial x} + \frac{\partial Q_\phi}{\partial \phi} + N_\phi &= 0 \end{aligned}$$

$$a \frac{\partial M_{x\phi}}{\partial x} - \frac{\partial M_{\phi}}{\partial \phi} - a Q_{\phi} = 0$$

$$\frac{\partial M_{x\phi}}{\partial \phi} + a \frac{\partial M_x}{\partial x} - a Q_x = 0$$

$$a(N_{x\phi} - N_{\phi x}) + M_{x\phi} = 0$$

(ii) Equation of geometry

$$\epsilon_x = \frac{\partial u}{\partial x} \quad ; \quad \epsilon_{\phi} = \frac{1}{a} \left(\frac{\partial v}{\partial \phi} - w \right) \quad ; \quad \gamma_{x\phi} = \frac{\partial u}{a \partial \phi} - \frac{\partial v}{\partial x} \quad ;$$

$$K_x = \frac{\partial^2 w}{\partial x^2} \quad ; \quad K_{\phi} = \frac{\partial^2 w}{a^2 \partial \phi^2} \quad ; \quad K_{x\phi} = \frac{\partial^2 w}{a \partial x \partial \phi} \quad ;$$

(iii) Stress-strain relation

$$N_x = Eh \epsilon_x \quad ; \quad N_{\phi} = Eh \epsilon_{\phi} \quad ; \quad N_{x\phi} = \frac{Eh}{2} \gamma_{x\phi} \quad ;$$

$$M_x = -DK_x \quad ; \quad M_{\phi} = -DK_{\phi} \quad ; \quad M_{x\phi} = -DK_{x\phi}$$

(iv) Donnell's equation

Substitute stress-strain relation into equation of geometry then to the equations of statics

$$a^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \left(\frac{\partial^2 u}{\partial \phi^2} + a \frac{\partial^2 v}{\partial x \partial \phi} \right) = 0 \quad \text{--- (i)}$$

$$\left(\frac{\partial^2 v}{\partial \phi^2} - \frac{\partial w}{\partial \phi} \right) + \frac{1}{2} \left(a \frac{\partial^2 u}{\partial x \partial \phi} + a^2 \frac{\partial^2 u}{\partial x^2} \right) = 0 \quad \text{--- (ii)}$$

$$\left(\frac{\partial v}{\partial \phi} - w \right) - \frac{h^2}{12a^2} \left(a^4 \frac{\partial^4 w}{\partial x^4} + 2a^2 \frac{\partial^4 w}{\partial x^2 \partial \phi^2} + \frac{\partial^4 w}{\partial \phi^4} \right) = 0 \quad \text{--- (iii)}$$

Apply operators $a \frac{\partial^2}{\partial x \partial \phi}$, $\frac{\partial^2}{\partial \phi^2}$ and $\left[a^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2} \right]$ into (i), (ii) & (iii) respectively. The equations reduced to

$$\left[a^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2} \right]^4 w + \frac{a^4}{k} \frac{\partial^4 w}{\partial x^4} = 0$$

where

$$k = \frac{h^2}{12a^2}$$

(v) Solution - Again, the solution is of the form

$$w = A e^{m\phi} \cos \frac{\lambda_n x}{a}$$

The characteristic equation is

$$(m^2 - \lambda_n^2)^4 + \frac{\lambda_n^4}{k} = 0$$

which has 8 roots

$$m_1 = \alpha_1 + i\beta_1 = -m_5$$

$$m_2 = \alpha_1 - i\beta_1 = -m_6$$

$$m_3 = \alpha_2 + i\beta_2 = -m_7$$

$$m_4 = \alpha_2 - i\beta_2 = -m_8$$

A general displacement form

$$w = \left[e^{-\alpha_1 \phi} (A_n \cos \beta_1 \phi + B_n \sin \beta_1 \phi) + e^{-\alpha_2 \phi} (C_n \cos \beta_2 \phi + D_n \sin \beta_2 \phi) \right] \cos \frac{\lambda n x}{a}$$

Expressions for stress resultants and displacements can be deduced from this in the usual way.

Remark on Donnell's theory

Donnell's theory may be termed 'exact' compared with methods examined so far in the sense that it does not admit any stress resultant and in that it is applicable to shells of any proportions. It is also the simplest of all other 'exact' solutions. Hoff¹⁵ and Kempner¹⁶ have investigated the accuracy of Donnell's equation by comparing its solution with that of Flügge's equation. They found fairly good agreement over a wide range of dimensions and loads. However Hoff reported that when the length of the cylinder is much greater than the diameter, the results are inaccurate. The A.S.C.E. manual on cylindrical shell roofs considers the theory as accurate enough for short shells whose span to radius ratio $\frac{L}{a} \leq 1.6$.

Donnell derived his equation in the study of elastic stability of cylinder¹³; the equation can be alternatively derived as a special case of the general theory of shallow shells.

1.7 Conclusion:

It has been shown that two dimensional shell theory is an approximate theory. Approximations have been introduced at almost all stages of analyses. The validity of such analysis might be questionable when there are so many approximations and they are not always consistent. For example transverse shear strains are neglected while transverse shear stresses are not. Unfortunately the same argument may be applied to most of structural theory (e.g. elementary beam theory uses the same approximation). The only possible justification for these approximations is experimental evidence. The analyses have been accepted mainly because they have produced satisfactory designs in practice..

The methods of analysis presented above give adequate tools for designing common shell structures, i.e. if the shell has no irregularities, & the load is uniformly distributed. If the shell has irregular boundaries, cut-outs...., or

Conclusion: (Cont'd)

if the load is concentrated or irregular, then the present analytical theory is either not capable of providing a closed form solution or the solution is so complicated that it is not suitable for usual design office procedures.

Thus there is a need for -

Rapid, simple experimental methods, so that σ measurements can be made, thereby enabling the use of model tests as a method of design and the formulation of simpler model for analysis.

Numerical methods to deal with situations which analytical methods are not able to handle efficiently.

Quick manual methods for checking purpose.

These problems will be dealt with in subsequent chapters.

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CHAPTER II

MOIRE LIGTENBERG METHOD APPLIED

to

CYLINDRICAL SURFACE

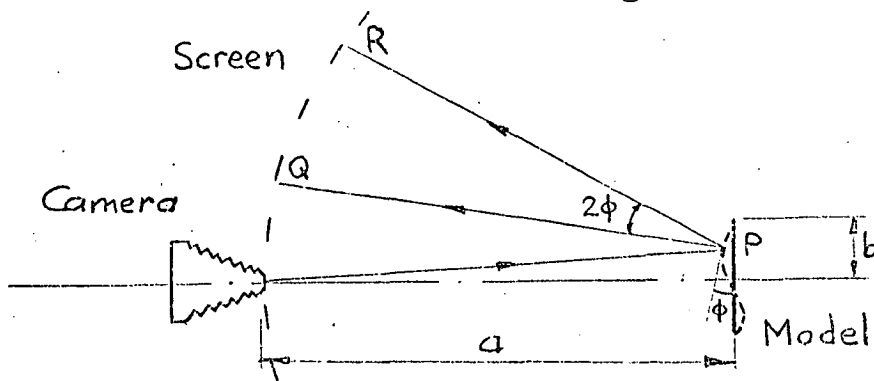
2.1 Introduction:

The analytical methods for solving the shell bending problems are long and complicated; moreover they cannot handle any irregularities that might occur, such as non-rectangular shapes, cut-outs, or concentrated loads. There is a need for obtaining a good insight into the flexural actions of shells so that approximate methods can be developed, based on simpler models. In this chapter, an experimental method of recording contours of slopes in the longitudinal and transverse directions of cylindrical surfaces is investigated. The method is an optical one based on the phenomenon of Moire fringes and it is an extension of the Ligtenberg method for flat plate.

The Moire Phenomenon - A Moire interference pattern is obtained by crossing two slightly different systems of distributed lines. For the Ligtenberg method, a ruled screen is placed in front of a model having a reflective surface; a camera is located behind a small hole in the screen. The reflection of the screen on the model gives a pattern of lines which is recorded by the camera. An exposure is first made with the model unloaded or with a small initial load; then the full load is applied, a second exposure is made on the same negative. The interference of the two images produces Moire fringes.

2.2 A brief summary of Ligtenberg method for flat plates:

- (a) Essential features of fringe formation.



The slope ϕ of a point P on the plate is determined by measuring the distance RQ on the screen. If the screen is flat, then

$$QR = 2\phi \sqrt{a^2 + b^2} = 2\phi a \sqrt{1 + \frac{b^2}{a^2}}$$

a = distance from screen to model.

b = model half width.

where b/a is the field of view of the apparatus and it depends on the dimensions of the plate and its distance from the screen.

If the model is small and kept far away from the screen then $\frac{b}{a} \ll 1$ hence $QR \simeq 2a\phi$.

Provided that the ruling on the screen is equispaced, all points which have the same amount of change of slope will have their images on the film displaced by the same amount RQ and create a fringe. Thus the fringe is a curve of constant slope on the plate. This slope is measured in the direction perpendicular to the directions of screen ruling.

It is desirable to maintain the simple relation $QR = 2a\phi$ as basis of fringe formation. For this relation to be exact, one needs to have $PQ=PR=a$ for all points P on the plate. It is seen at once that this is not possible, however with a judicious choice of screen shape, the variation of the 'reflected light paths' can be kept to a minimum.

(b) Shape of screen for flat plate -

The correct shape of screen for flat plate can be obtained either graphically or analytically. The main criteria is that the reflected light path must be approximately constant. Ligtenberg has solved the problem completely in his paper. He suggested that a circle of radius 3.5 to 4 times the distance from screen to model gives the least error in slope (for a field of view $\frac{b}{a} = 0.4$ the error is less than 0.3%). He also suggested a range of pitches of ruling of screen for satisfactory fringe formation (about 11 lines per inch).

2.3 Extension of the method for cylindrical surface:

Ligtenberg method can be extended to cover the case of cylindrical shell. For simplicity, we will confine ourselves to circular shells, however the method is equally applicable for any other cylindrical shell shape.

(a) Graphical derivations -

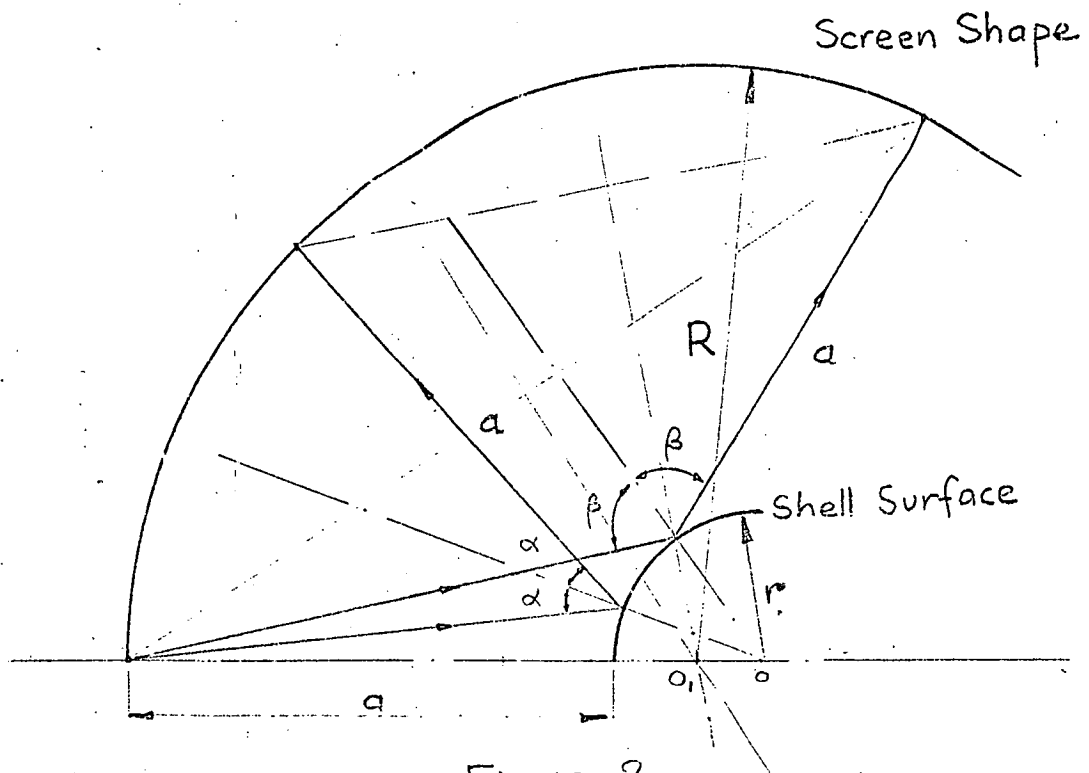
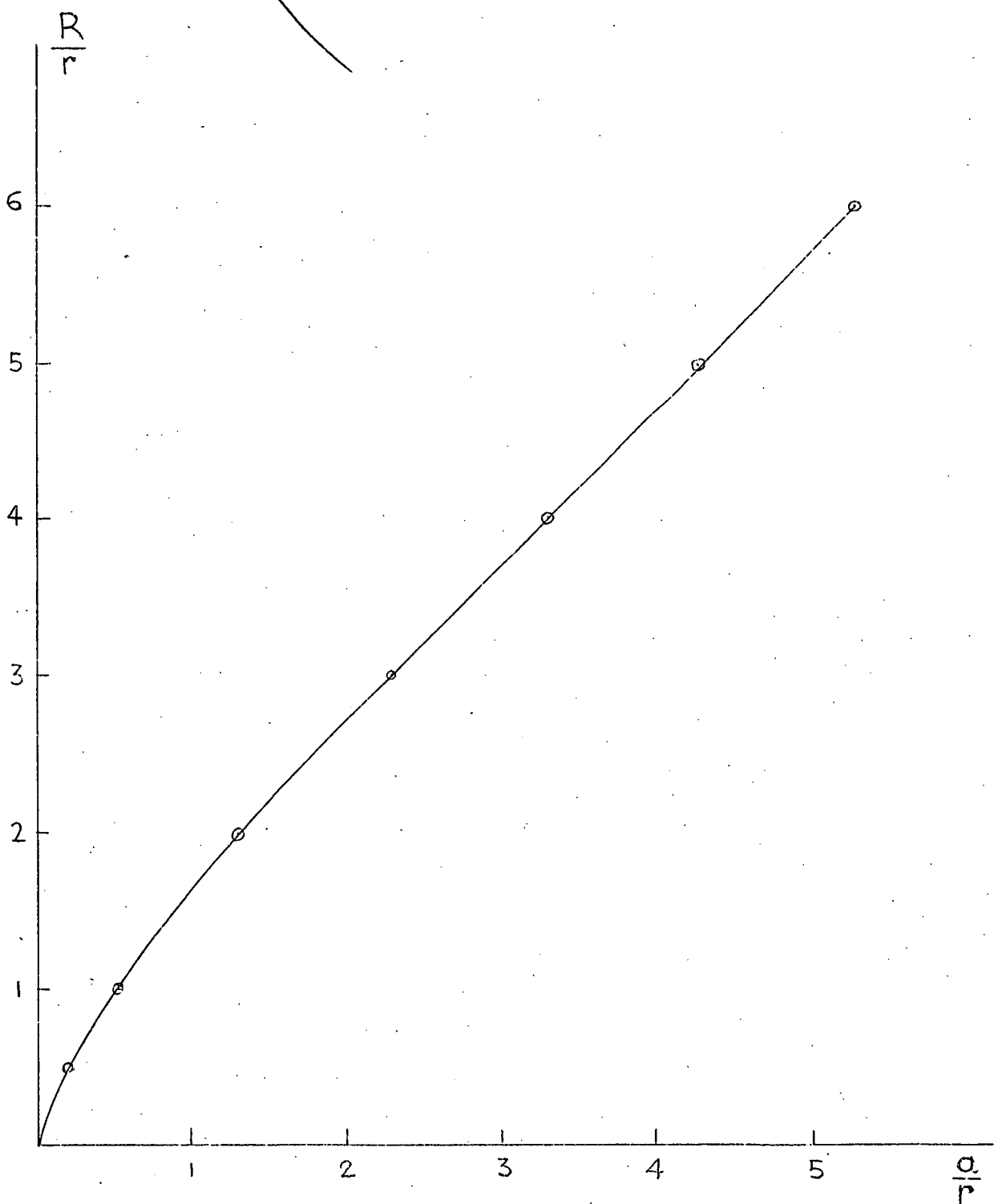
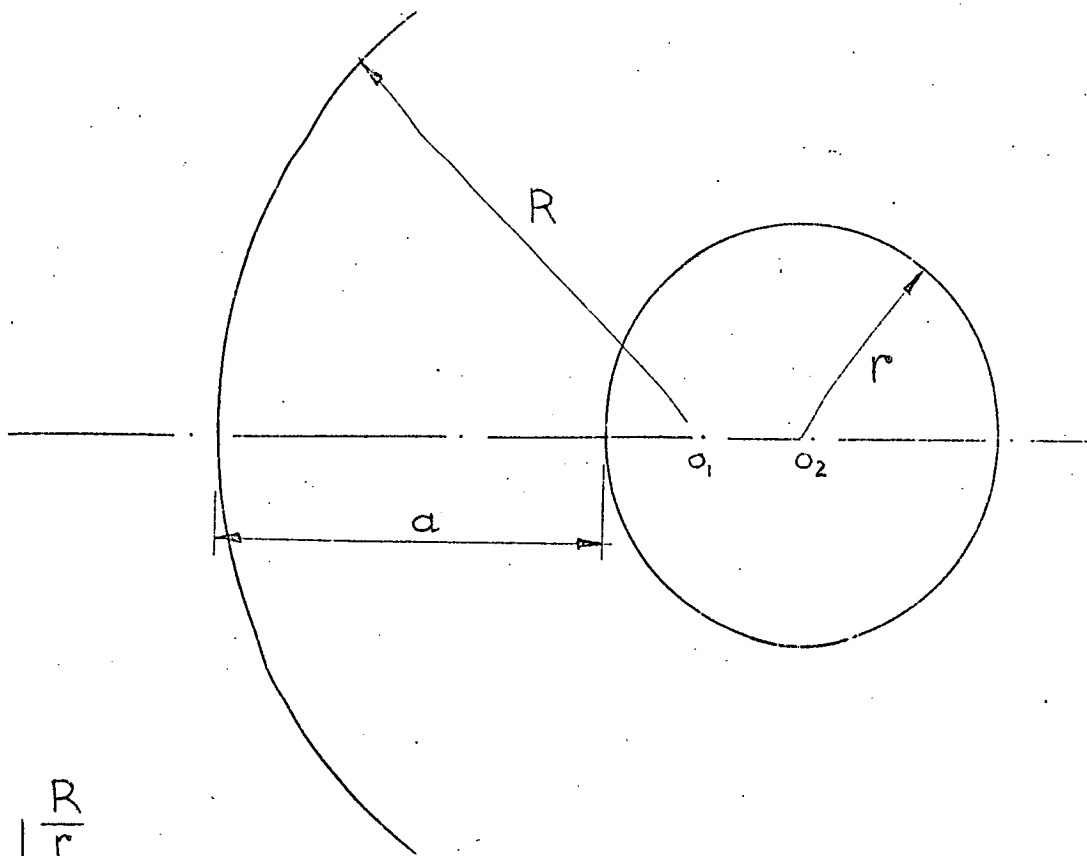


Figure 2.

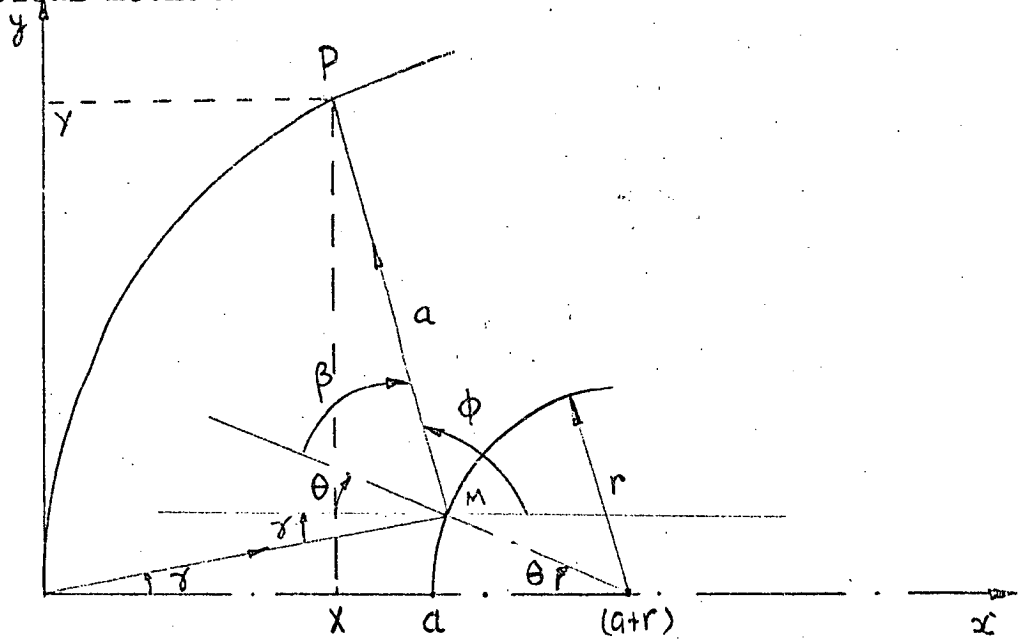
For a given size of model and distance from screen to model. The correct shape of screen can be determined by ray tracing for a number of points on the model surface as shown in Figure 2.

From various ray tracing exercises based on the principle of constant reflected light paths, it was concluded that a circle can be fitted reasonably well through the points defining the screen shape. The centre of the screen is offset to that of the shell. The three main variables of this problem are the screen radius ' R ', the model radius ' r ' and the 'light path' ' a '. Given any two of these, the third quantity can be calculated. Therefore it is necessary to construct only one screen, this screen can be used for shells of various radii by placing the shell at the appropriate distance. All the information can be condensed into one single graph by plotting $\frac{R}{r}$ against $\frac{a}{r}$ (see graph 1). It is noted that the relation between $\frac{R}{r}$ and $\frac{a}{r}$ becomes nearly linear for $\frac{a}{r} > 2$.



Graph 1

(b) Analytical method.



The reflected light path 'a' is given. The shell is circular, a point M on the shell surface is specified by the angle θ . The camera sees the point M at the angle γ .

If β is the reflected angle then $\beta = \gamma + \theta$

Coordinate axis are set up from centre of camera.

Let ϕ be the angle of the reflected ray and

$$\phi = \pi - 2\beta + \gamma = \pi - 2\theta - \gamma.$$

The position of M is given by $x = a + r - r\cos\theta$.

$$y = r\sin\theta.$$

Camera angle is given by $\gamma = \arctan \frac{r\sin\theta}{a + r - r\cos\theta}$.

The position of P is given by:

$$X = a + r(1 - \cos\theta) + a\cos\phi.$$

$$Y = r\sin\theta + a\sin\phi.$$

A computer program may be written, following the above analysis, to obtain a suitable shape for the screen. The use of the analytical method is not recommended since it is much quicker to solve the problem graphically.

2.4 Experimental Set-up:

(a) The Screen - A screen was constructed following the above consideration. A sheet of paper ruled with lines of constant pitch was glued to a flexible aluminium sheet to represent the screen. The screen was fixed to a steel frame to form the desired circular shape. The distance between camera and model was calculated so that an adequate field of view was available. The ruling on the screen was made to a pitch of two line per inch and five line per inch. The

reduction of pitch due to convex surface is then easily coped with. For a model of 10" radius placed 20" from the camera, a pitch of 5 lines/inch is just distinguishable with naked eye and a pitch of 2 lines/inch gives good contrast.

Lighting - Two systems of lighting were tried

- (i) Circular neon tube (100 watts) gives uniform lighting but contrast is not very good because the light is too diffused and because of insufficient intensity.
- (ii) Standard light globe (500 watts) gives better contrast but lighting easily becomes uneven if the lights are not adjusted properly.

Since the shell surface is convex it is not possible to put the whole surface into sharp focus, therefore it was decided to choose a position that put the whole picture into a good average sharpness so that the rulings are distinguishable everywhere.

Kodak photomechanical films Ortho type 3 are used, they give satisfactory results on low aperture opening ($\frac{1}{64}$) and long exposure time (1 - 2 minutes).

(b) The model - Any material that can be formed into circular shell shape and can be made reflective on one side may be used for the model. In forming the shell shape, mechanical or heat treatment processes are usually employed; therefore it is important that these processes should not alter the physical properties of the material too much, otherwise it will be difficult to determine the physical constants associated with the model.

Relative merits of stainless steel model and plastic model are now discussed.

(i) Stainless steel models have the advantages of

- Fairly uniform thickness, this is very helpful since any variation in thickness will affect the flexural rigidity $D = \frac{Eh^3}{12(1-\nu^2)}$ and it has been shown that these variations affect the shell's behaviour very considerably.

- Constant physical properties; variations in Young's modulus and Poisson's ratio are usually small, the material is fairly isotropic.

- No tendency to creep.

But they also have the following disadvantages

- High stiffness which will require large load to cause recordable bending deformation (as shown in chapter I bending deformation is only secondary deformation for shells).

- A quick model analysis will show that the material is not suitable for modelling shell roofs from prototype structures, e.g. concrete shell with radius 30' span 120' thickness 4" - Model is to have a radius of 10" span 40" then the thickness will be 0.33". This thickness is too excessive for practical loading to cause any bending deformation.

(ii) Perspex model - Perspex has a lower Young modulus and therefore it is more suitable to model shell roofs from prototype structures. The greater thickness also allows larger deformation and still conforms to small deflection theory. However, perspex tends to creep and its physical properties may vary severely from sheet to sheet.

To make a shell model it is necessary to form the material into circular shape. For steel, this can easily be done on a rolling machine without greatly altering its thickness. For plastic, a heat treatment is required to soften the plastic, this will considerably alter the uniformity of its thickness.

Various other materials can be used for modelling such as brass plate, painted steel, etc. It was decided to use stainless steel model for the preliminary experiments since it was simplest to make and a great deal of information is still obtainable with these models.

2.5 Relation between photographic picture and actual shell surface:

Due to the convexity of shell surface, the relation between the photographic picture and the shell surface is no longer simple. Certain distortions occur and they must be clearly recognised. It is also important to distinguish between the original shell slope and the slope of the deformed shape. The fringes only measure the change of slope.

(i) In the longitudinal direction - There is a slight distortion in the longitudinal direction. This is shown as a slightly curved line on the photograph for a straight line on the model (see Fig. 4). The reason for this is simple (see Fig. 5).

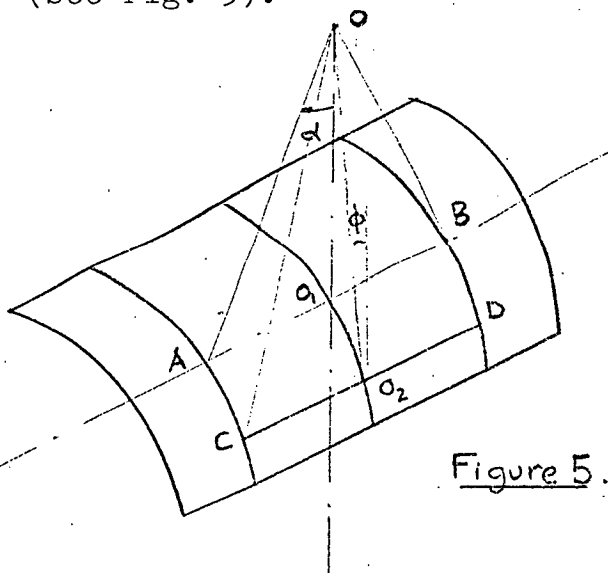


Figure 5.

Consider two equal lengths AB and CD on the model.

As seen from camera O

$$AB = 2 OO_1 \tan \alpha$$

$$CD = 2 OO_2 \tan \alpha$$

Where $OO_1 = OO_2 \cos \phi$.

As ϕ increase OO_2 increases

Thus, on the photograph, CD is longer than AB; therefore

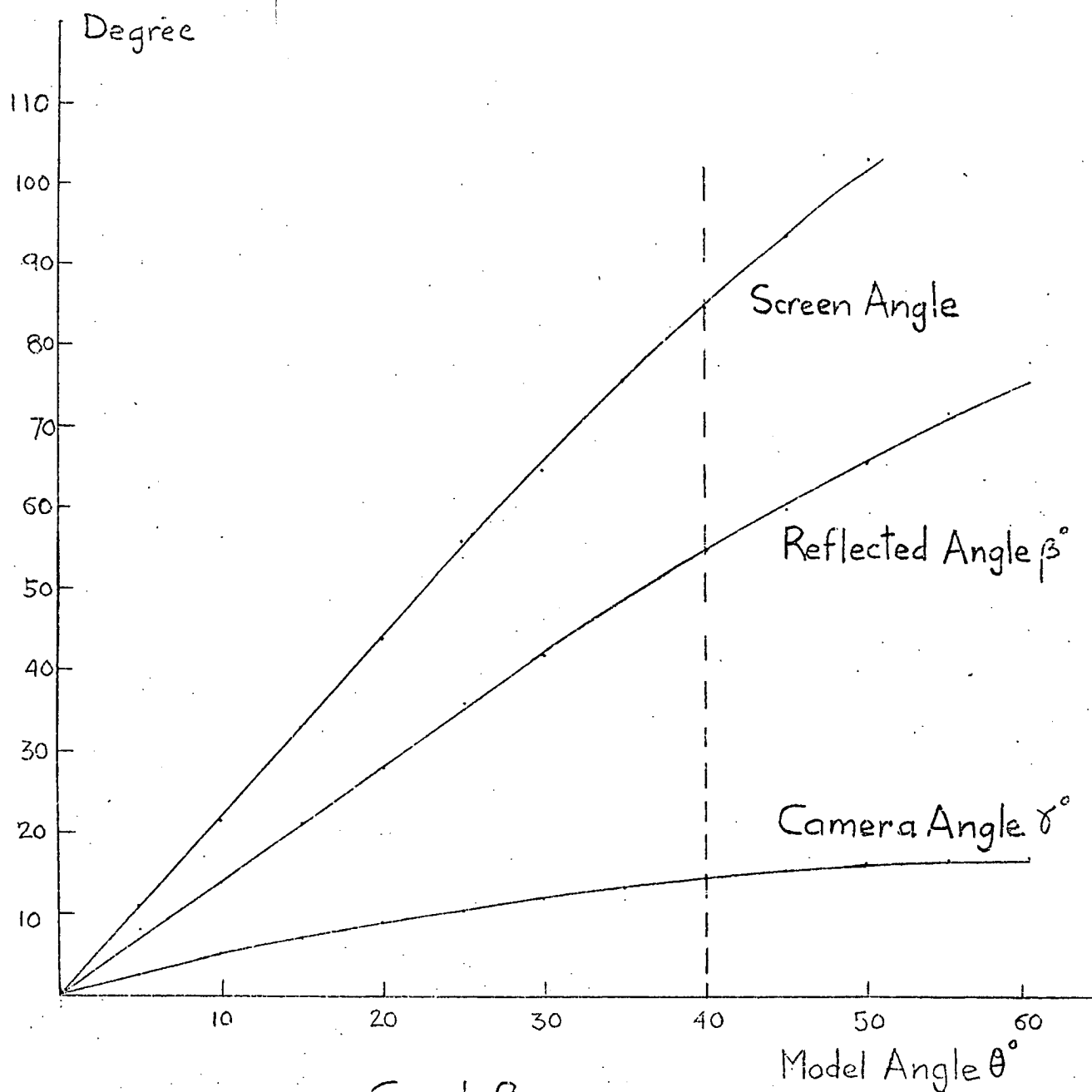
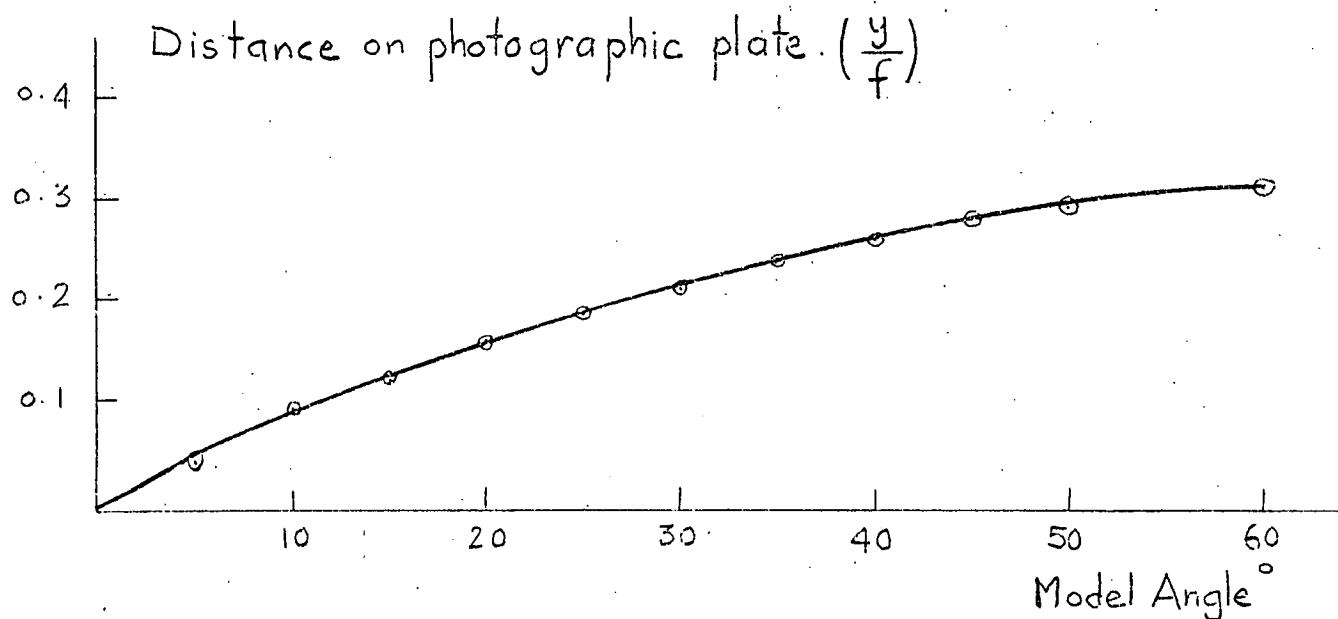
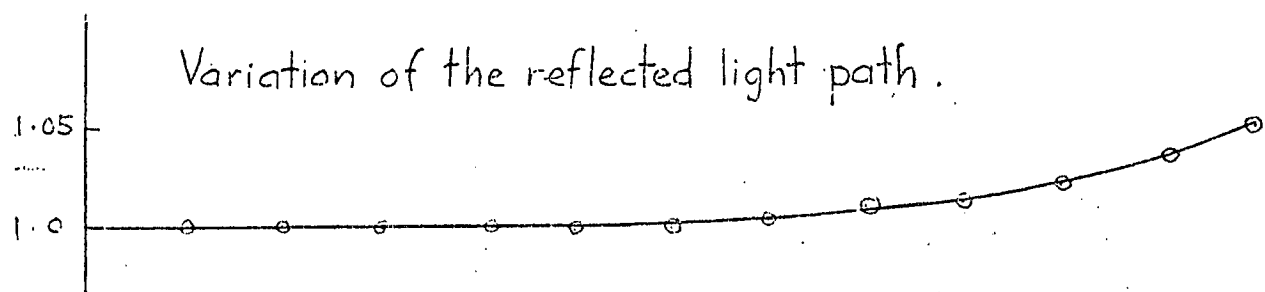
while O_1O_2 is straight on the photograph, AC and BD are curved. This distortion is quite small but it varies with the field of view and the distance from camera to model.

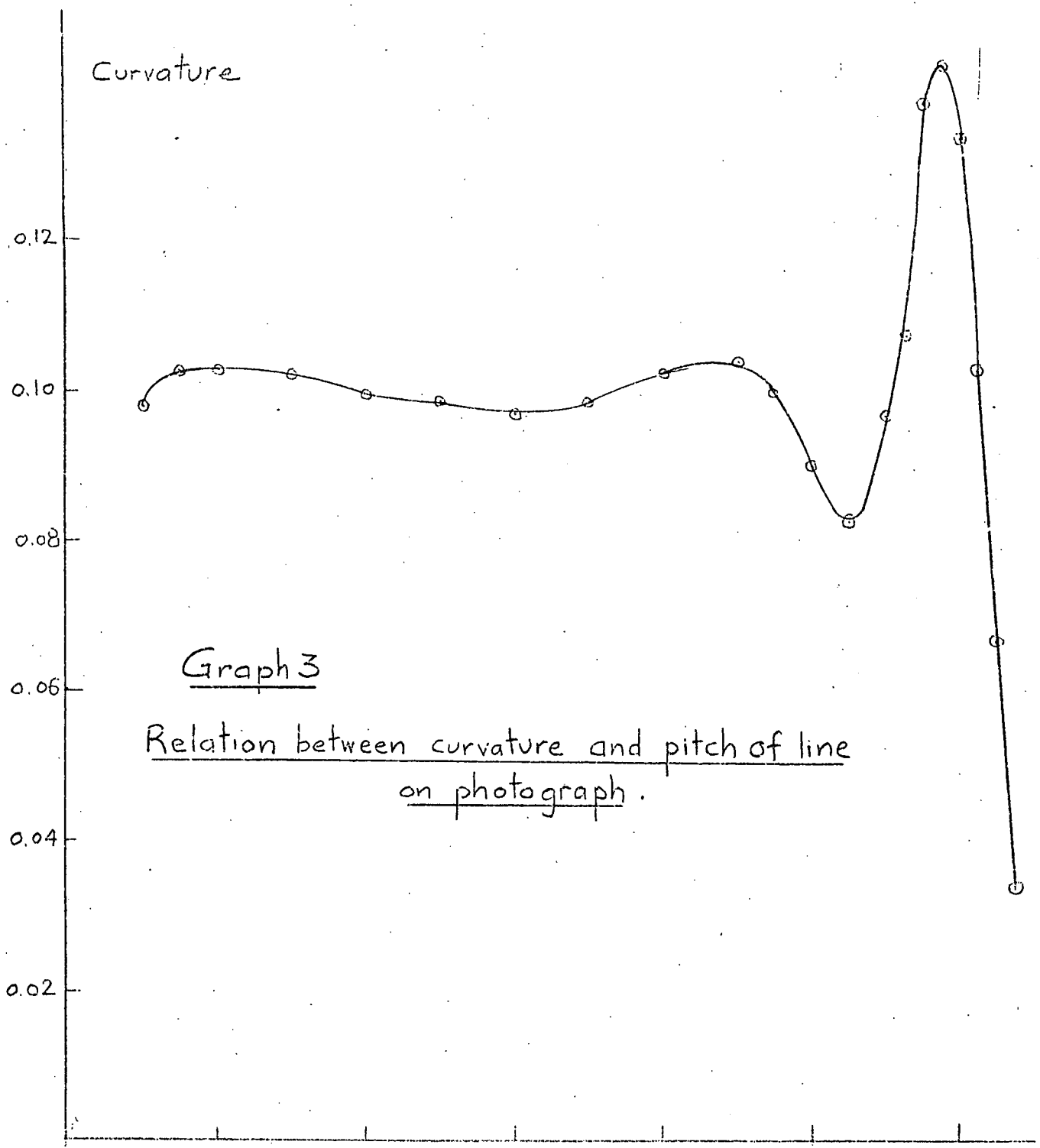
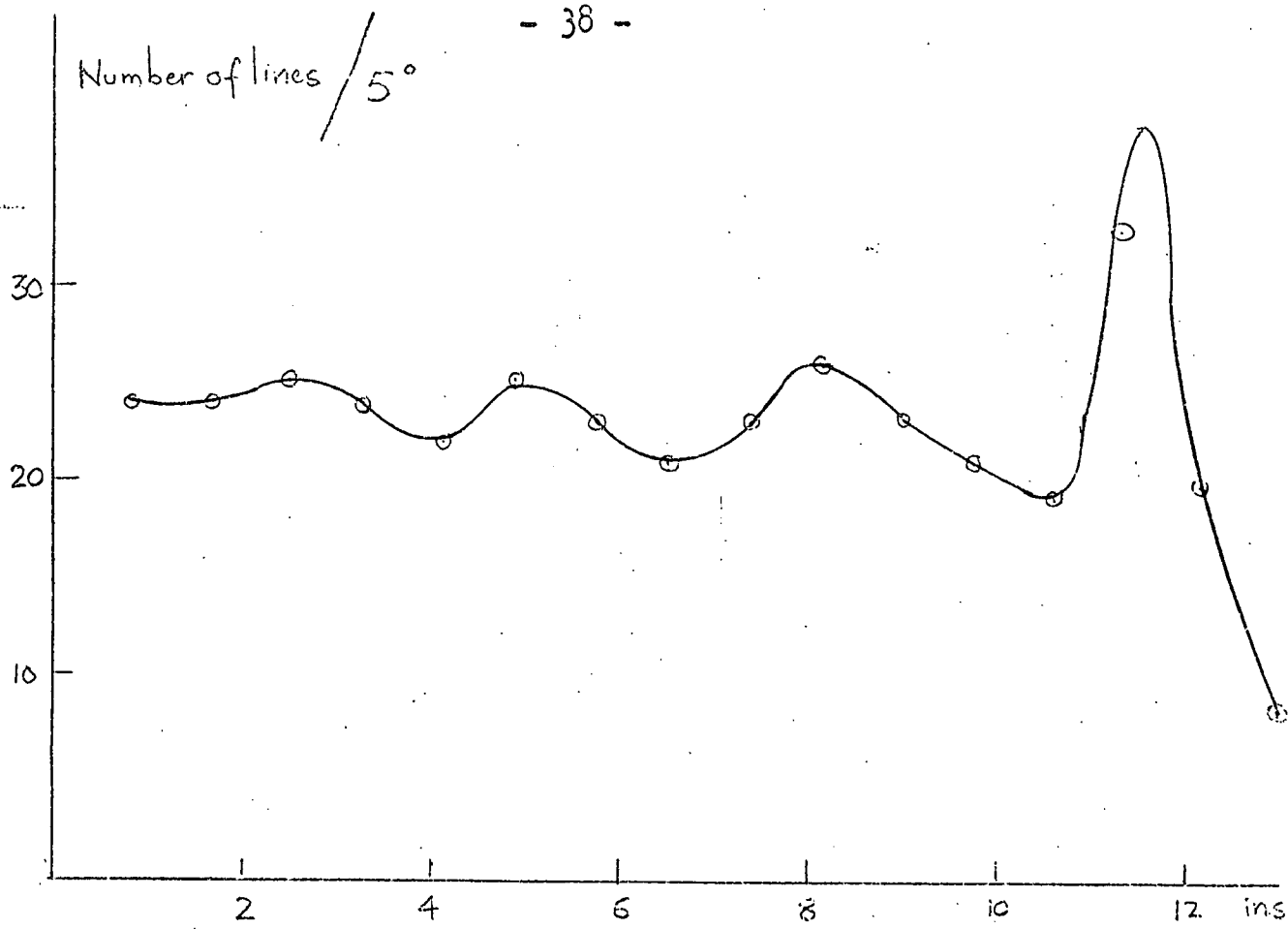
(ii) In the transverse direction - There is a reducing effect in the transverse direction due to the convex surface. The image of the screen on the photograph has varying pitch since it is a projection of the curved shell surface on a plane. When this variation is accounted for, the number of lines per inch of the image of the screen on the model is approximately constant.

Graph 2 - obtained by ray tracing - describes the relations between various angles involved and it is seen that most of the relations are approximately linear for the range of shell angle $0-40^\circ$ (within 5%).

(iii) Use of the photographs as a means of measuring the shell curvature -

If the surface is smooth and of constant curvature then the number of lines per inch of shell is approximately constant.





Thus the pitch of the lines on the photograph is a measure of the curvature of the shell. If the curvature varies, experiments show that the pitch of the lines on the photograph also varies. Graph 3 shows the relation between the pitch of the lines on the photograph, and curvature of a shell measured by another means. It is seen that the apparatus can be used as a curvature meter after some calibrations. A single exposure photograph shows the variation of the curvature of the shell.

(iv) Fringes created by rigid body motion - A minor drawback of Ligtenberg method applied to cylindrical shells is that a rigid body rotation about the vertical axis of the shell produces fringes.



Figure 6

Fringes are also obtained with a translation movement as shown in Fig. 5. As both types of fringes correspond to displacements which do not strain the shell, they must be allowed for to obtain a correct picture of the actual shell deformations.

(v) Errors in measurements of slopes due to radial and tangential displacements

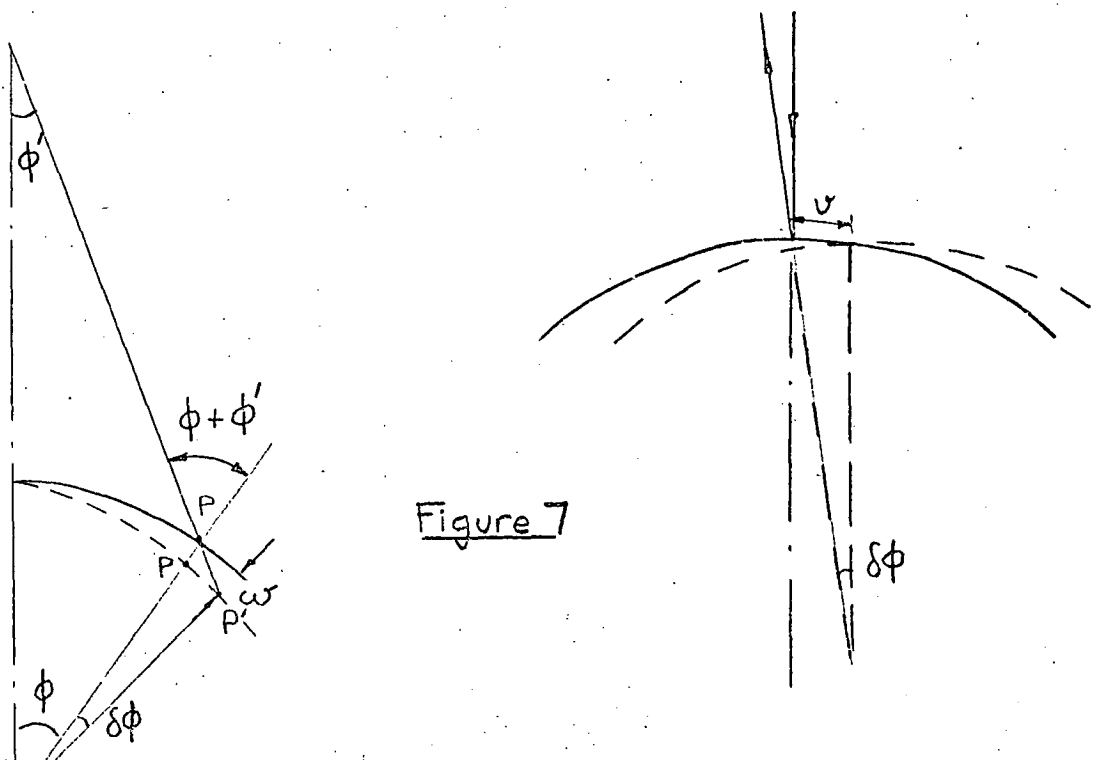


Figure 7

Radial and tangential displacements cause errors in measurements of slope. These errors can be evaluated and allowed for.

A radial displacement w at shell angle ϕ causes an angular displacement as seen from camera

$$\delta\phi = \frac{w \tan(\phi + \phi')}{a}$$

If the points P and P₁ are sufficiently close so that the slope change due to the loading of shell is approximately equal, then the photograph records an extra amount of slope change $\delta\phi$. The effect of radial displacement in slope measurement is most pronounced near the shell edge and is very small near the crown.

A tangential displacement v similarly causes an angular displacement recorded on photographic plate

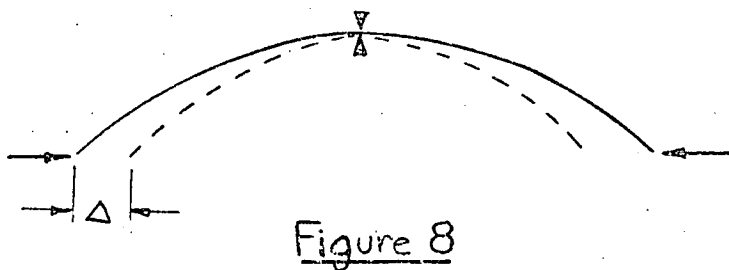
$$\delta\phi = \frac{v}{a} \quad (\text{See Fig. 7})$$

Both of these errors could be substantial and should be corrected to obtain a better measurement of slope.

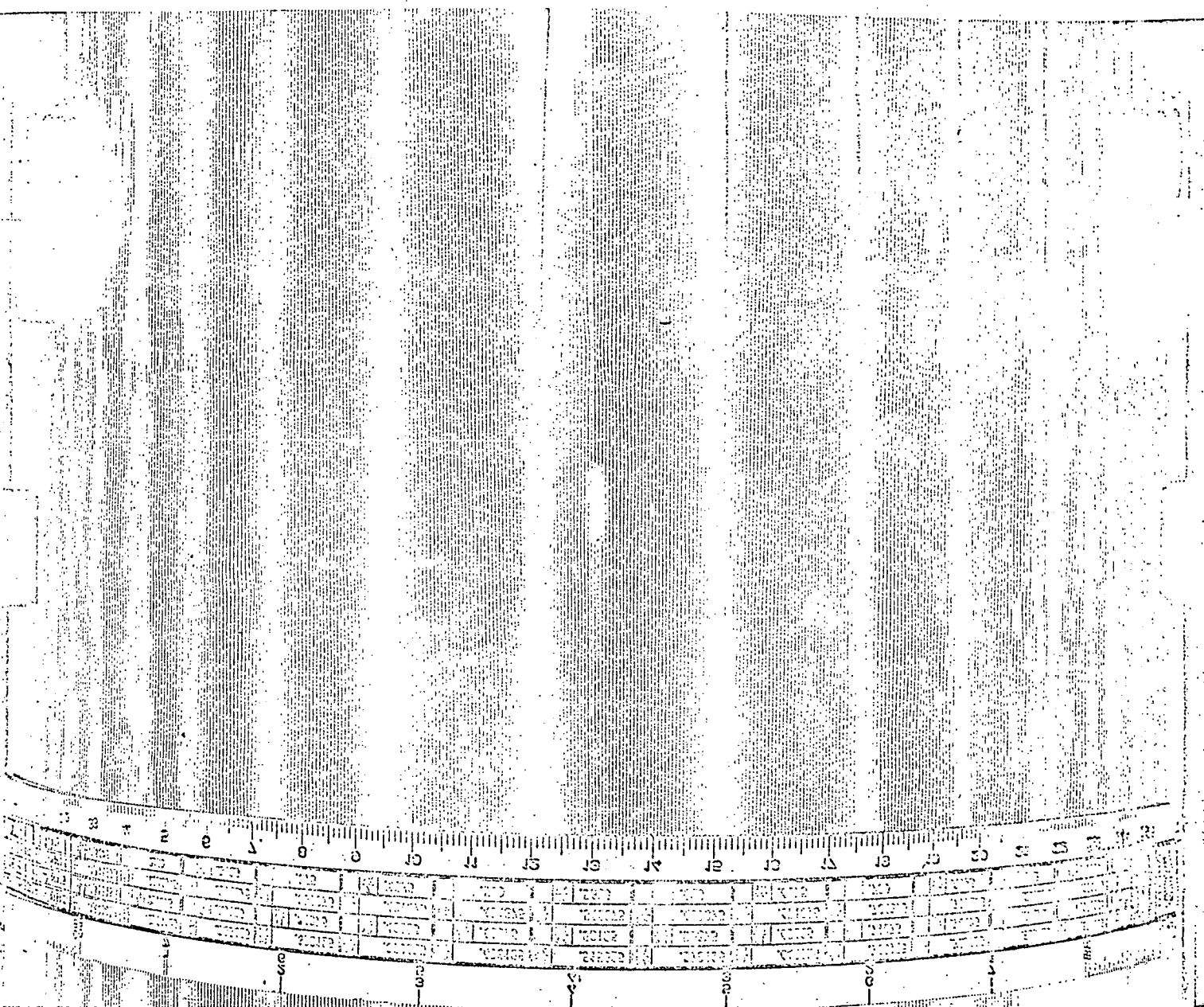
2.6. Some simple experiments:

The following experiments were carried out to test the above measuring technique.

(a) Experiment No. 1 - This experiment was designed to test the accuracy of the technique for measuring slope in transverse direction.



The model shell was given a known amount of displacement by applying a concentrated load at the edge (see Fig. 8). The crown of the shell was prevented from moving. The bending of the shell is largely inextensional.



y	$d\theta$	$yd\theta$	$\sum yd\theta$	ϕ°	$\frac{w}{a} = \frac{u \sin \phi}{a}$	$\delta d\theta$	$d\theta'$	$yd\theta'$
1.60	0.0033	0.0053	0.0053	7			0.0033	0.0053
1.40	0.0099	0.0138	0.0191	16			0.0099	0.0138
1.13	0.0165	0.0182	0.0373	25	0.001	0	0.0165	0.0185
0.82	0.0231	0.0185	0.0558	32	0.005	0.0045	0.0186	0.0152
0.56	0.0297	0.0162	0.0720	37	0.007	0.0078	0.0220	0.0123
0.46	0.0363	0.0160	0.0880	40	0.010	0.0128	0.0235	0.0109
0.28	0.0429	0.0120	0.1000	42	0.012	0.0170	0.0259	0.0073
0.23	0.0495	0.0113	0.1113	43	0.014	0.0206	0.0295	0.0068
0.12	0.0561	0.0067	0.1180	44	0.015	0.0230	0.0331	0.0040
0.08	0.0627	0.0050	0.1230	45	0.016	0.0256	0.0371	0.0030

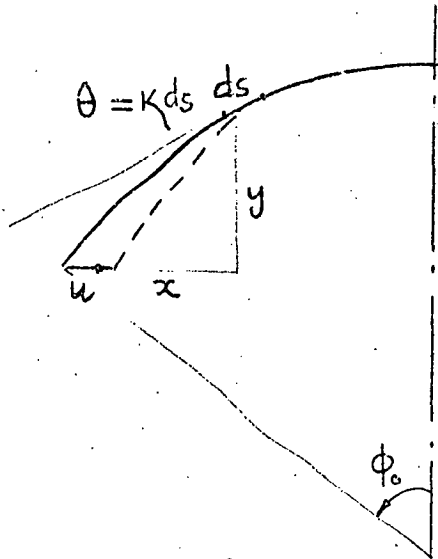
Total horizontal displacement $u = 0.123''$ (uncorrected) 23%
 $= 0.097''$ (corrected) 3%

(c.f. dial gauge measurement $u = 0.100''$)

Figure 9.

By the technique of double exposure, fringes were obtained. All the fringes were straight confirming that they are indeed lines of constant slope. (Fig. 9)

The amount of movement at the edge can be obtained from the fringe measurement as follows:-



Consider only half of the shell (symmetry). Let the curvature of an element ds be changed by an amount K .

Let u be the horizontal displacement,

$$du = K ds = y d\theta$$

Total horizontal displacement is obtained by integration.

$$u = \int_0^{\phi_0} K y ds = \int_0^{\phi_0} y d\theta$$

The total horizontal displacement

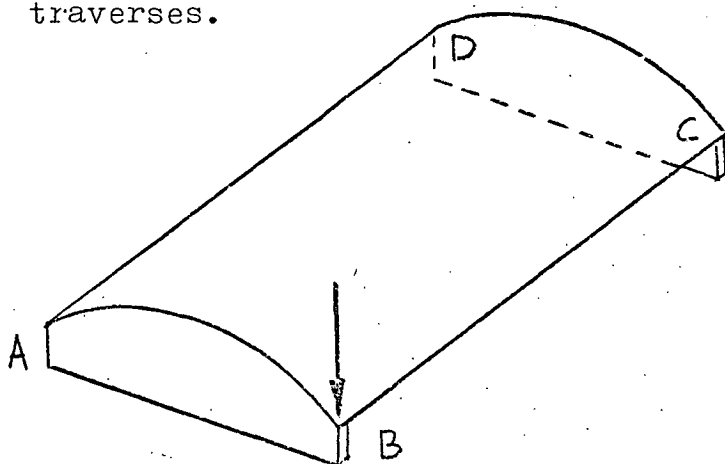
is measured with dial gauge. This causes considerable radial displacement and the slope values of the fringes near the edge are corrected with the formula

$$\phi = \frac{\omega \tan(\phi + \phi')}{a}$$

derived in the previous section.

Fig. 9 presents the fringe pattern and the tabulated results. It is seen that the radial displacement correction for slope is important in obtaining accurate measurements.

(b) Experiment No. 2 - Twisting of a shell having end traverses.

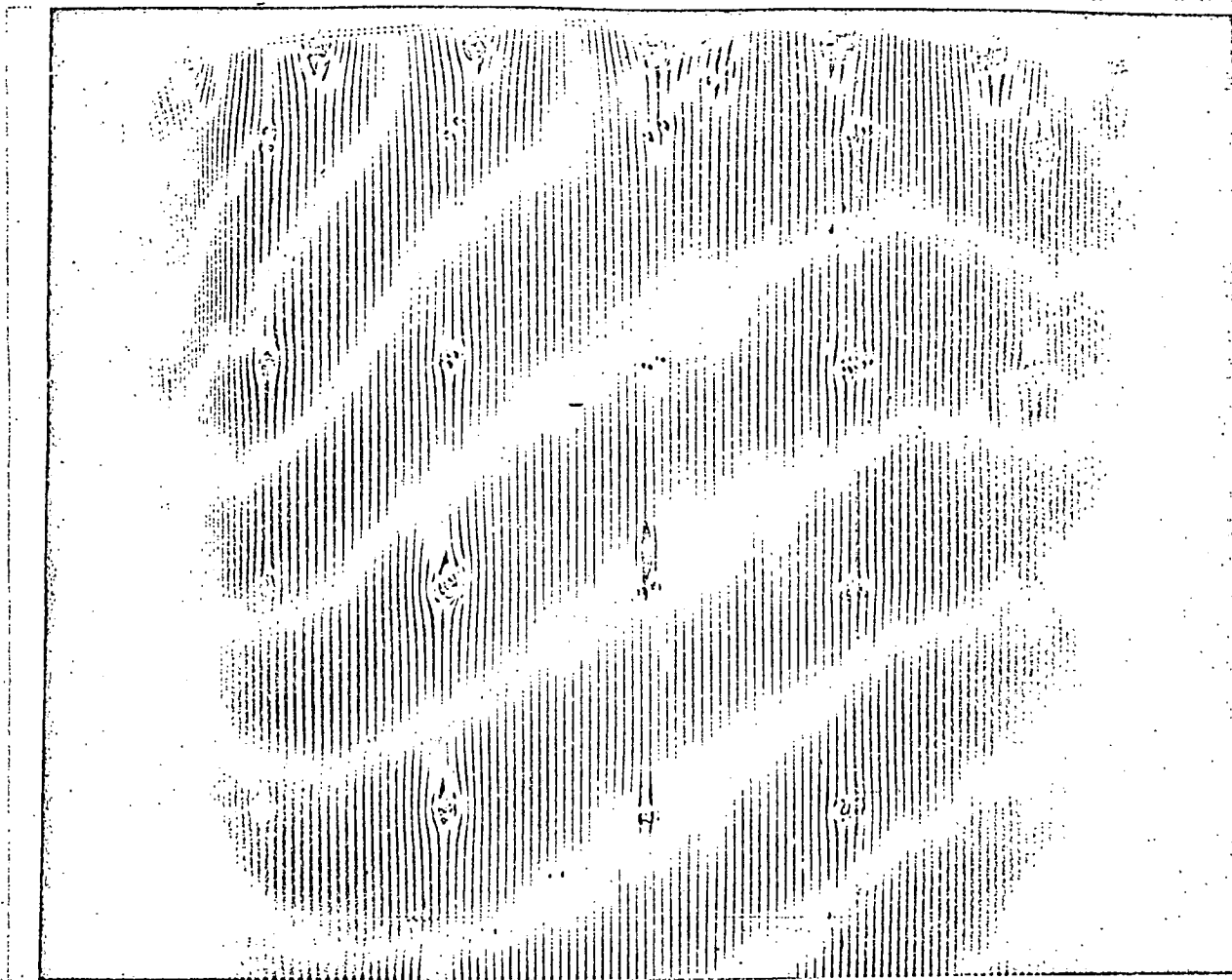


The shell was supported at A, C and D. Load was applied to B. From symmetry all corner reactions are equal in magnitude. Vertical deflection at B was measured with a dial gauge.

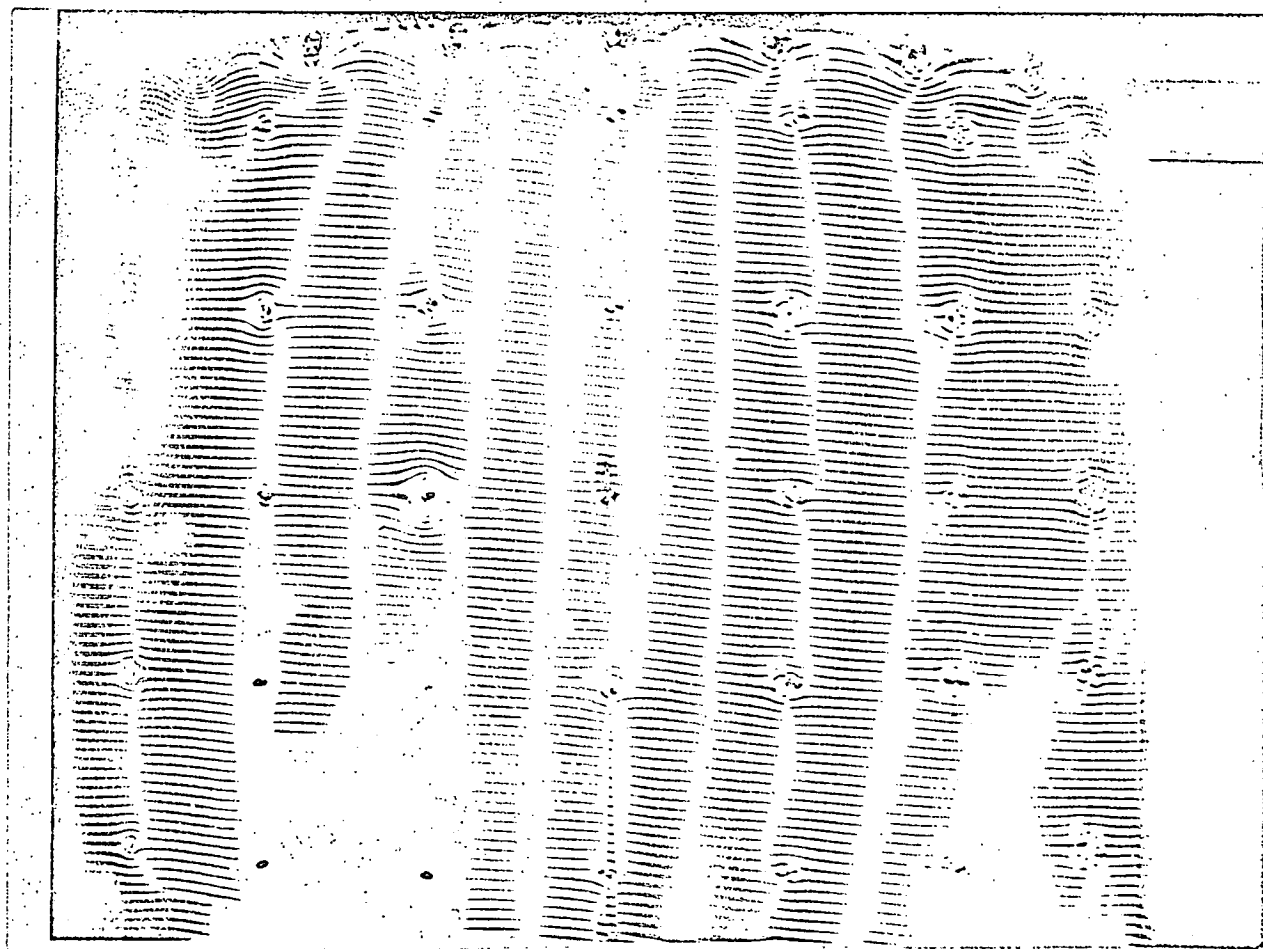
Photographs were taken with screen ruling in both directions. (See Fig. 10).

It was found that the application of the load inadvertently rotated the shell about the vertical axis. These displacements were also measured with dial gauges.

CONTOURS OF SLOPES DUE TO TWISTING OF SHELL.



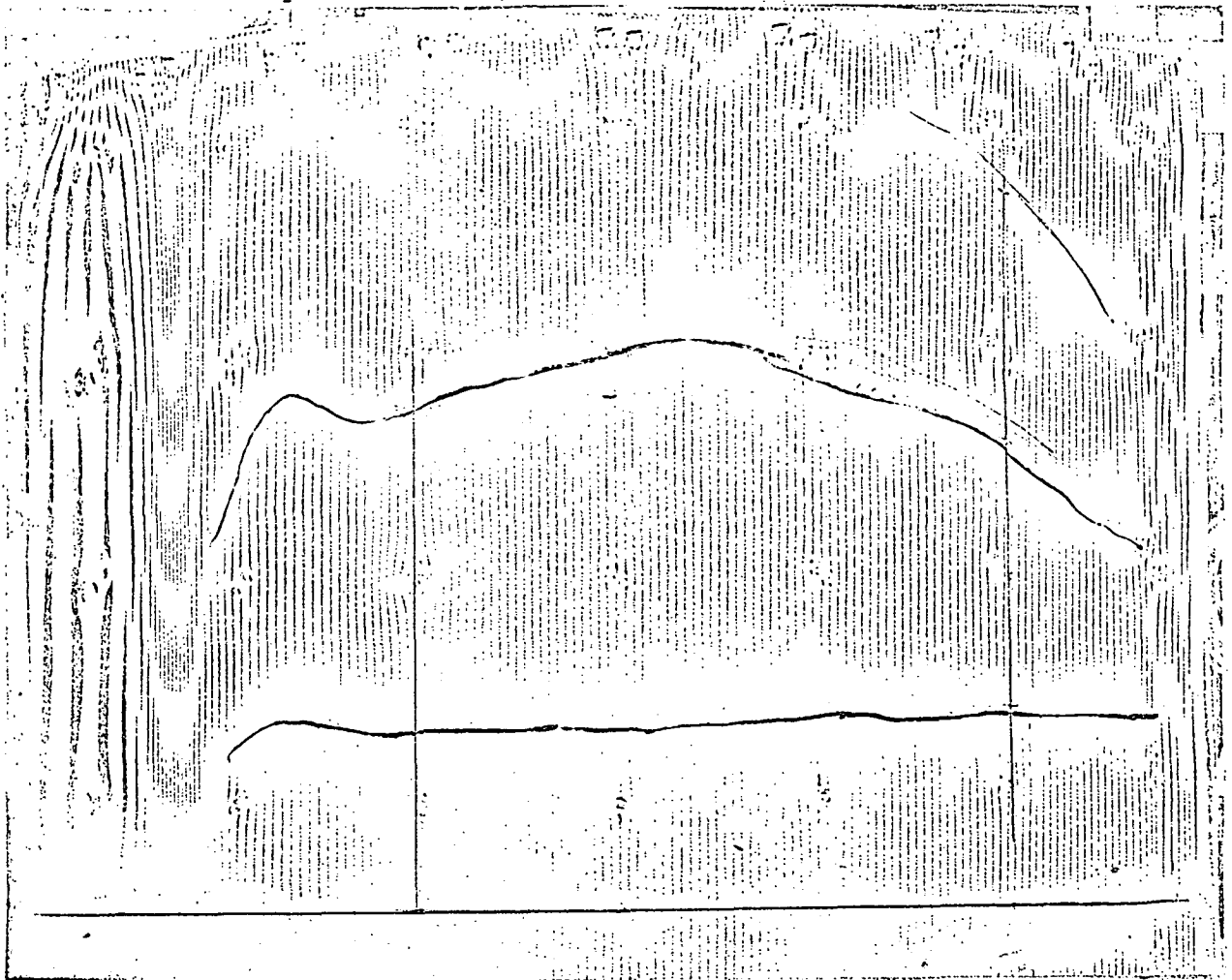
SLOPE IN TRANSVERSE DIRECTION



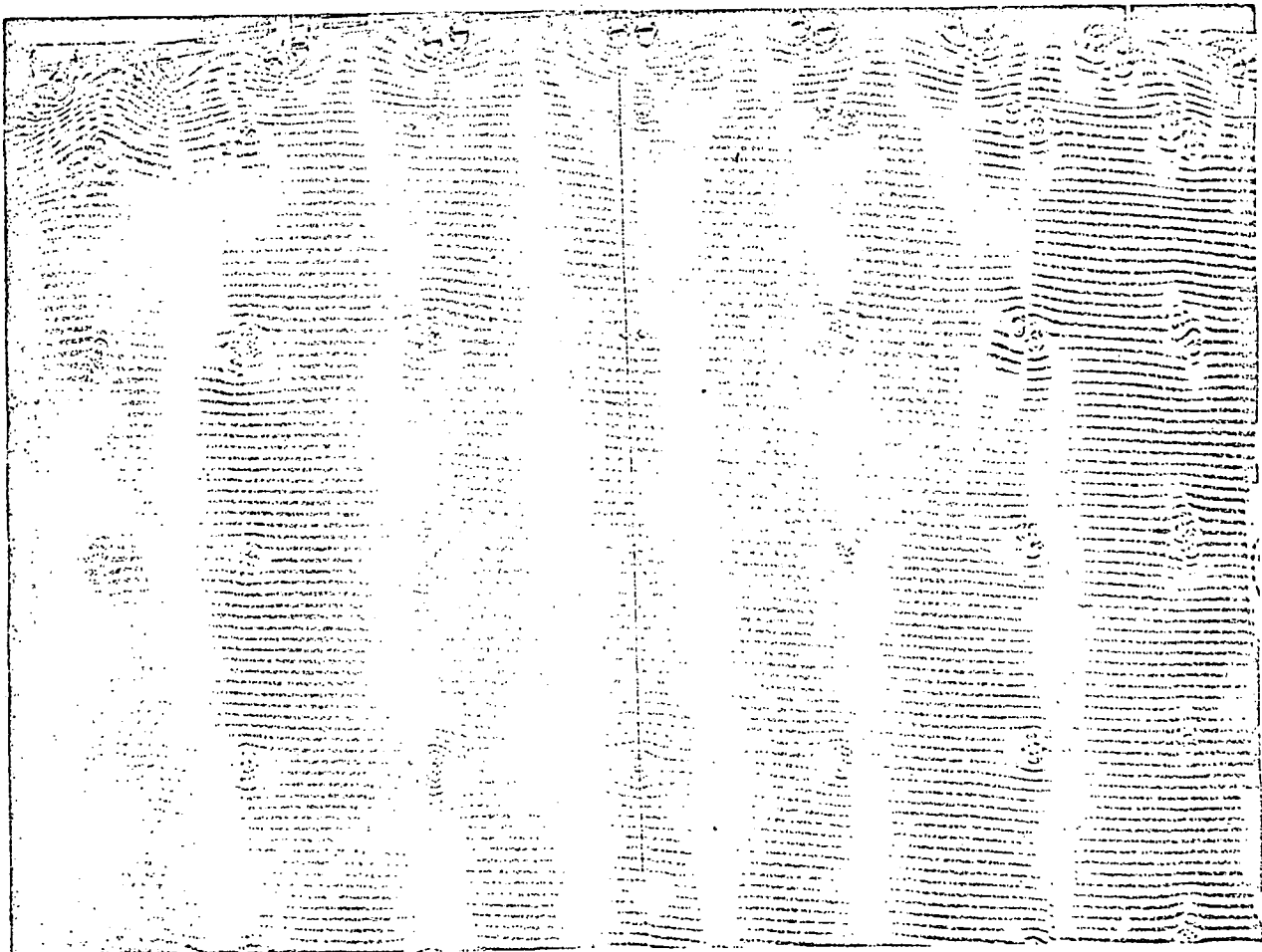
SLOPE IN LONGITUDINAL DIRECTION

FIGURE 10(a)

CONTOURS OF SLOPES DUE TO ROTATION OF SHELL



SLOPE IN TRANSVERSE DIRECTION



SLOPE IN LONGITUDINAL DIRECTION

Slope in longitudinal direction

0.03

0.02

0.01

△

△

△

△

△

Uncorrected Line

Corrected Line

$$\frac{\partial^2 \omega}{\partial \phi \partial x} = \frac{0.008}{10 \times 0.179} = 0.0046$$

ϕ°

40

30

20

10

Along transverse direction.

Slope in transverse direction

0.06

0.05

0.04

0.03

0.02

0.01

$\phi = 0^\circ$

$\phi = 23^\circ$

Corrected line

$$\frac{\partial^2 \omega}{\partial x \partial \phi} = \frac{0.009}{2} = 0.0045$$

7

6

5

4

3

2

1

0

1

2

3

4

5

6

7

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Graph 4

(i) Calculation of twist from dial gauge measurements.

Displacement in x direction $u = 0.22''$.

Displacement in y direction $v = 6.25''$.

Displacement in z direction $w = 0.50''$.

Twist due to w (assuming constant twist over the whole surface).

$$\frac{\partial w}{\partial x} = \frac{2}{2 \times 16} = 0.062 \quad ; \quad \frac{\partial^2 w}{\partial x \partial \phi} = \frac{0.062}{14} = 0.0044 / \text{in}.$$

(ii) Calculation of twist from photograph.

The photograph also record the slope change due to the rigid body rotation about its vertical axis. This must be corrected by taking a photograph of slope change due to rotation only and subtracting it from the first photograph. The results are plotted in Graph 4 .

From measurement of $\frac{\partial w}{\partial x}$, the twist is 0.0046/in.

" " " $\frac{\partial w}{\partial \phi}$, the twist is 0.0045/in.

The maximum difference between the three results is less than 5%.

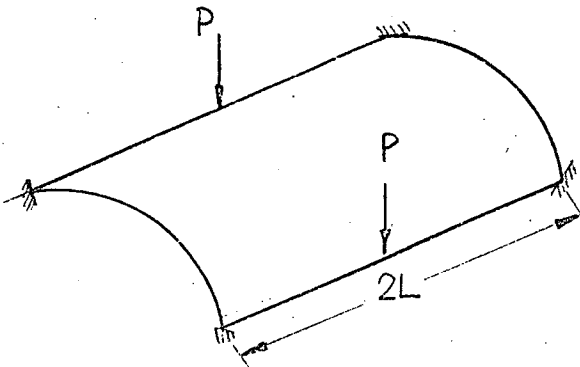
(iii) Some comments on the photographs of contours of slope.

Figure 10 presents photographs of contours of slope obtained by twisting and rigid body rotation. The contours of slope in longitudinal direction are straight lines. This suggests that straight line in the longitudinal direction remains straight after twisting the shell. Contours of slope in transverse direction are anti-symmetrical about the longitudinal axis.

2.7. A shell problem.

In this section, the problem of a concentrated load on a shell edge is investigated. The aim is to show the use of the new experimental technique in solving shell problems especially in conjunction with energy methods.

(a) Description of the problem.



Details of the model:

Material: Stainless Steel.

Thickness $h = 0.023''$.

Young modulus $E = 30 \times 10^6$ psi.

Length $2L = 8''$.

Radius $a = 5''$.

Shell Angle $\phi_0 = 36^\circ$.

Load $P = 5$ lbs.

The shell was held rigidly at four corners. Loads were applied vertically at the center of the edges.

(b) Measurements.

The shell model was set up in the laboratory and measurements were made:

- Deflections were measured with 0.001" dial gauges.
- Strains along the edges were measured with Huggenberger mechanical strain gauges.
- Surface slopes were measured with the new technique described above.

The results are presented in Graph 6.

The measurements give the experimenter an appreciation of the problem and an understanding of the way the shell carries its load. These insights are valuable guides for the formulation of the mathematical problem. From measurements, the following observations are made:

- The shell carries its loads mainly by bending action. Considerable moments are recorded at regions around the supports and at the point of application of the load, in both transverse and longitudinal directions.

- Deformation of the shell is concentrated in a small region around the load. It varies rapidly in the transverse direction (exponentially perhaps), no definite trend is observed in the longitudinal direction.

(c) Solution by energy method.

The Rayleigh-Ritz potential energy method is used to solve the above problem. The use of energy is merely a means of satisfying the statical condition approximately. The process consists of choosing a plausible deformation function containing a number of free parameters. Variation of the potential energy expression allows the determination of the parameters in such a way that the equations of statics are satisfied on the average. The deformation function is chosen to satisfy as many boundary conditions as practicable and to describe the deformation pattern of the shell as closely as possible. The contours of slope obtained by the above technique offer a very good guide to the functional form of the deformation.

For the problem under discussion, measurements have shown that the bending action is most dominant. Thus, for a first approximation membrane stresses are neglected. Potential energy expression, for a quarter of a shell under bending action only, is written as:

$$U = a \int_0^{\phi_0} \int_0^{\frac{1}{2}} \left(\int_0^{K_x} M_x dK_x + \int_0^{K_\phi} M_\phi dK_\phi + \int_0^{K_{x\phi}} M_{x\phi} dK_{x\phi} \right) dx d\phi - \frac{1}{2} \int_0^\delta P d\delta.$$

If the deflection of the shell surface is written as

$$w = f(A_1, A_2, \dots)$$

where $A_1, A_2 \dots$ are some disposable parameters,

Variation of the energy expression with respect to $A_1, A_2 \dots$

gives a set of equations of the type

$$\frac{\partial U}{\partial A_1} = 0 \Rightarrow \frac{1}{2} P \frac{d\delta}{dA_1} = a \iint (M_x \frac{dK_x}{dA_1} + M_\phi \frac{dK_\phi}{dA_1} + M_{x\phi} \frac{dK_{x\phi}}{dA_1}) dx d\phi.$$

$$\frac{\partial U}{\partial A_2} = 0 \Rightarrow \frac{1}{2} P \frac{d\delta}{dA_2} = a \iint (M_x \frac{dK_x}{dA_2} + M_\phi \frac{dK_\phi}{dA_2} + M_{x\phi} \frac{dK_{x\phi}}{dA_2}) dx d\phi.$$

The equations of geometry are simplified to

$$K_x = \frac{\partial^2 w}{\partial x^2} ; K_\phi = \frac{1}{a^2} \frac{\partial^2 w}{\partial \phi^2} ; K_{x\phi} = \frac{1}{a} \frac{\partial^2 w}{\partial x \partial \phi} ;$$

The stress-strain relations are

$$M_x = D K_x ; M_\phi = D K_\phi ; M_{x\phi} = D K_{x\phi} ;$$

From these three sets of equations the values of $A_1, A_2 \dots$ can be determined for any given shape of w .

Using the shape

$$w = A_1 \left(\cos \frac{\pi x}{L} + A_2 \right) \cosh A_3 \phi.$$

A_2 is found through boundary condition $A_2 = 1$.

$$K_x = -A_1 \left(\frac{\pi}{L} \right)^2 \cos \frac{\pi x}{L} \cosh A_3 \phi.$$

$$K_\phi = \frac{A_3^2 A_1}{a^2} \left(\cos \frac{\pi x}{L} + 1 \right) \cosh A_3 \phi.$$

$$K_{x\phi} = \frac{A_3 A_1}{a} \frac{\pi}{L} \sin \frac{\pi x}{L} \sinh A_3 \phi.$$

$$\begin{aligned} \sum \iint M \frac{dK}{dA_1} &= DA_1 \left(\frac{\pi}{L} \right)^4 \frac{L}{4} \left(\phi_0 + \frac{1}{2A_3} \sinh 2A_3 \phi_0 \right) + \frac{3A_3^4}{4} \frac{DA_1 L}{a^4} \left(\phi_0 + \frac{1}{2A_3} \sinh 2A_3 \phi_0 \right) \\ &+ \frac{A_3^2}{4a^2} DA_1 L \left(\frac{\pi}{L} \right)^2 \left(\frac{1}{2A_3} \sinh 2A_3 \phi_0 - \phi_0 \right) = \frac{P}{a} \cosh A_3 \phi_0. \end{aligned}$$

$$\begin{aligned} \sum \iint M \frac{dK}{dA_3} &= A_1^2 A_3 D \left(\frac{\pi}{L} \right)^4 \frac{L}{2} \frac{1}{4A_3} \left(\cosh 2A_3 \phi_0 - 1 \right) + \frac{A_3^5 A_1^2 D}{4} \frac{3L}{8A_3} \left(\cosh 2A_3 \phi_0 - 1 \right) \\ &+ \frac{DA_3^3}{a^2} A_1^2 \left(\frac{\pi}{L} \right)^2 \left(\frac{L}{2} \right) \frac{1}{4A_3} \left(\cosh 2A_3 \phi_0 - 1 \right) = \frac{PA_3}{a} \sinh A_3 \phi_0. \end{aligned}$$

These two equations are solved for A_1 and A_3 .

$$A_1 = 0.006 ; A_3 = 4.2.$$

Values of deflection, slope, and curvature are plotted in Graph 6.

This solution, as seen in Graph 6, gives good agreement in deflection and slope values, however large discrepancies in curvatures and hence moments values are obtained. The reason is that the deformation function satisfies slope and deflection boundary condition but fails to observe the no-moment condition at the free edge.

It appears that higher derivatives must be satisfied to obtain a good moment distribution. It is also noted that the variation of stresses in transverse direction is similar to that of a beam on elastic foundation. From these considerations, the following deformation function is used :

$$w = A \left[-\left(x - \frac{L}{2}\right)^5 + \frac{5}{16} L^4 x - \frac{9}{32} L^5 \right] e^{-\lambda \phi} \sin\left(\lambda \phi + \frac{\pi}{2}\right).$$

$$\gamma_x = A \left[-20\left(x - \frac{L}{2}\right)^3 \right] e^{-\lambda \phi} \sin\left(\lambda \phi + \frac{\pi}{2}\right).$$

$$\kappa_\phi = \frac{2\lambda^2 A}{a^2} \left[-\left(x - \frac{L}{2}\right)^5 + \frac{5}{16} L^4 x - \frac{9}{32} L^5 \right] e^{-\lambda \phi} \sin \lambda \phi.$$

$$\kappa_{\phi\phi} = -\frac{5A}{a} \sqrt{2} \lambda \left[-\left(x - \frac{L}{2}\right)^4 + \frac{1}{16} L^4 \right] e^{-\lambda \phi} \sin\left(\lambda \phi + \frac{\pi}{4}\right).$$

λ is found through boundary condition consideration.

A is the solution of the following equation :

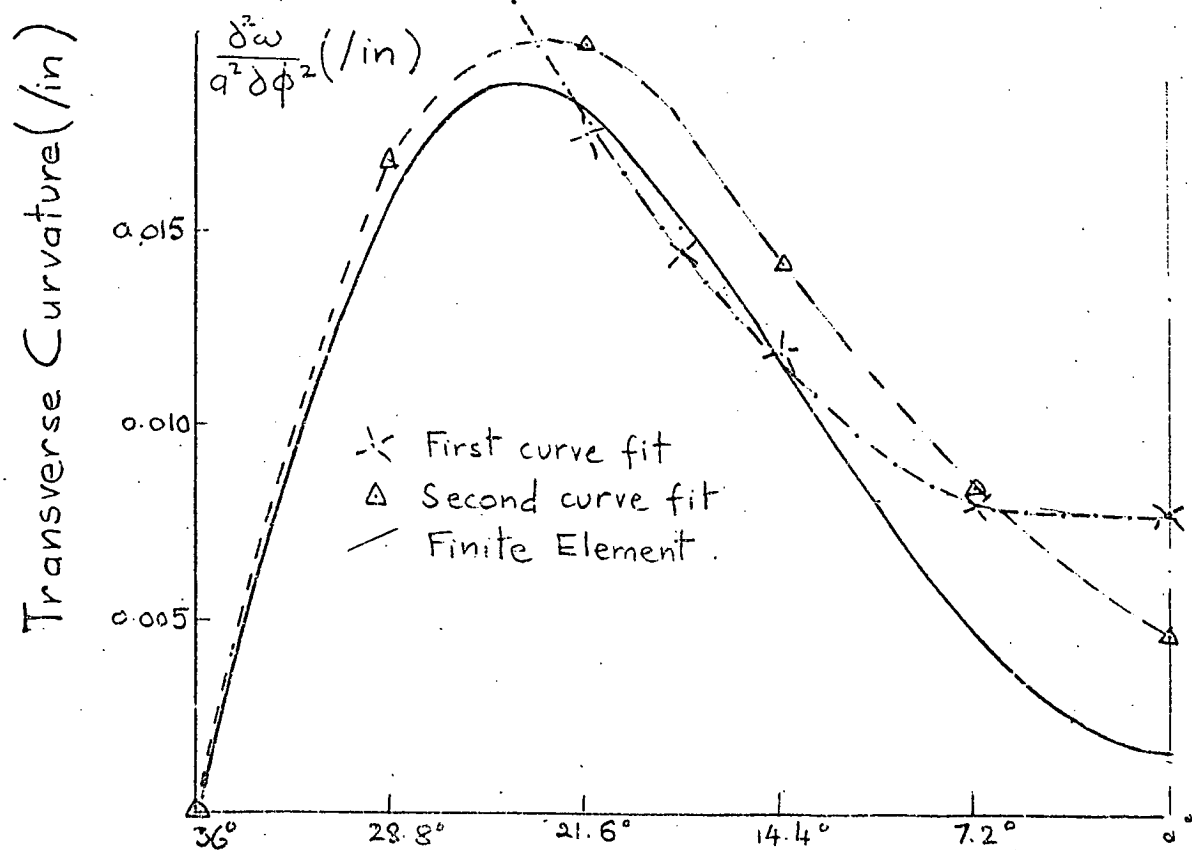
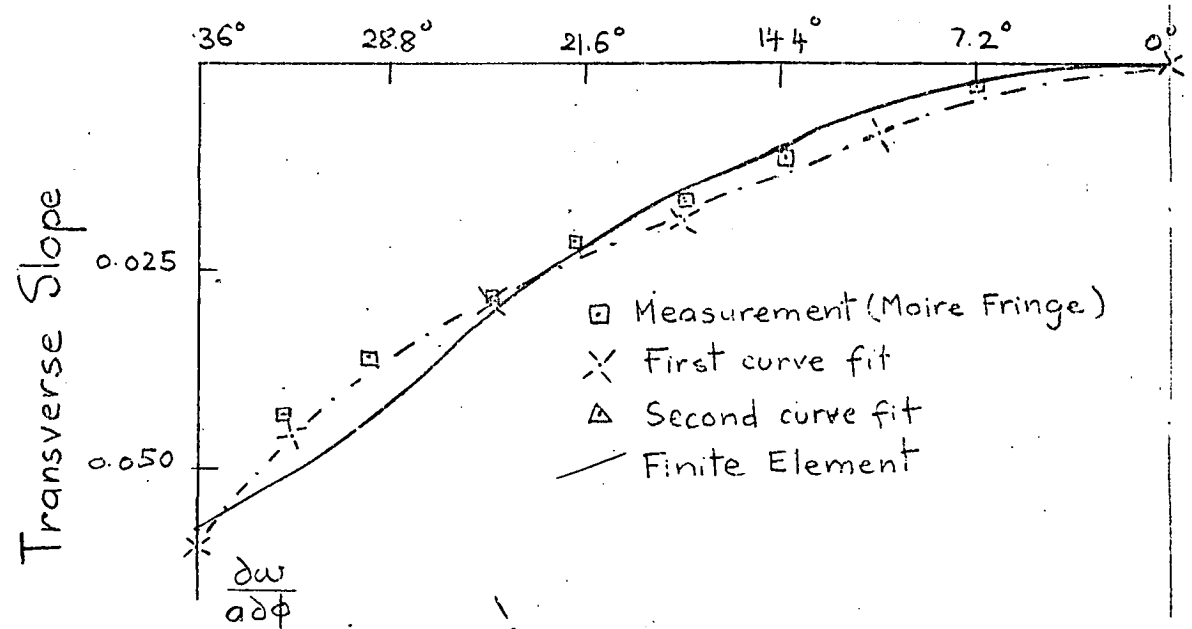
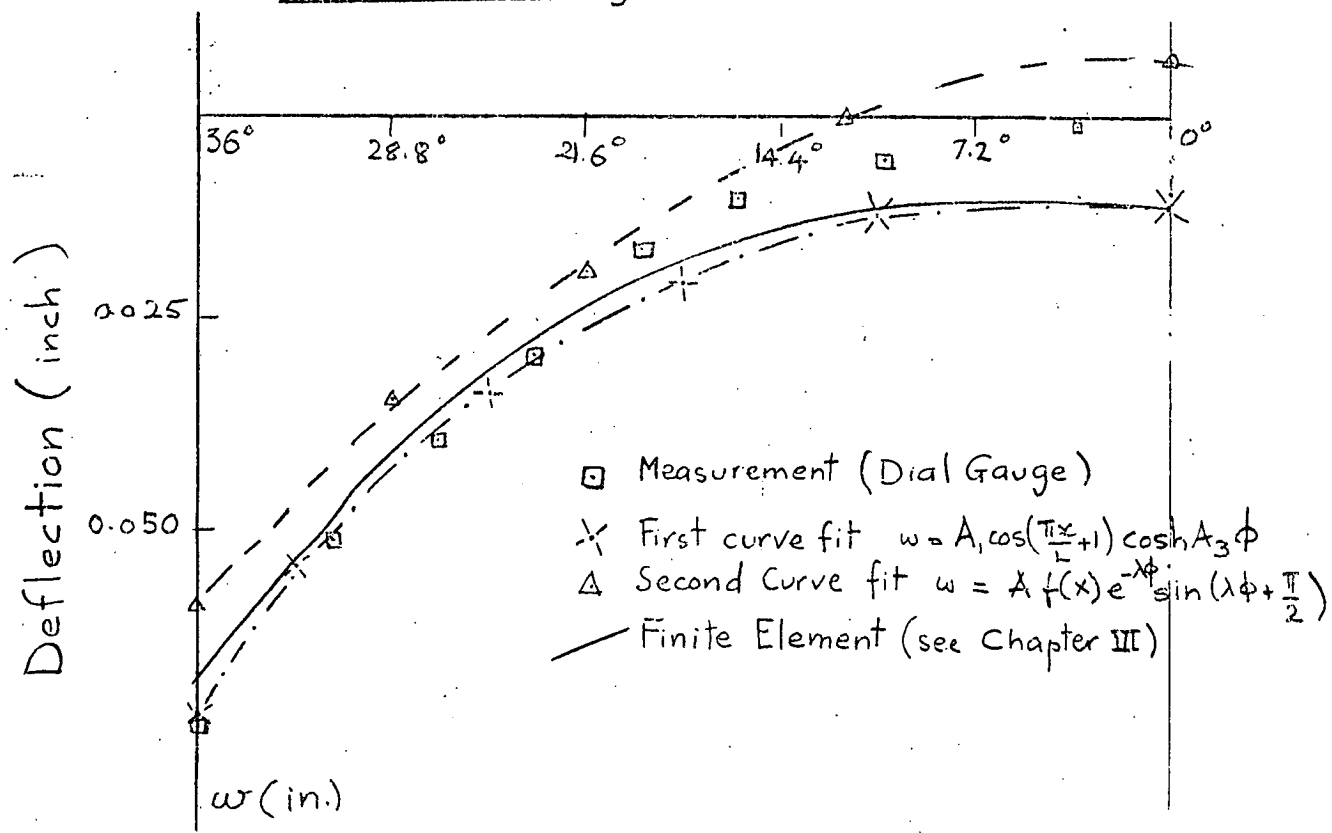
$$\begin{aligned} \sum \iint M \frac{dk}{dA} &= 400A \frac{2}{7} \left(\frac{L}{2}\right)^7 \frac{1}{2} \left\{ \frac{1}{2\lambda} + \frac{1}{2\sqrt{2}\lambda} \sin \frac{\pi}{4} \right\} \\ &+ \frac{4\lambda^4}{a^4} A \left(0.0236 L^8 - 2.678 \left(\frac{L}{2}\right)^8 \right) \frac{1}{2} \left\{ \frac{1}{2\lambda} - \frac{1}{2\sqrt{2}\lambda} \sin \frac{\pi}{4} \right\} \\ &+ \frac{L^9}{256} - \frac{52}{90} \left(\frac{L}{2}\right)^9 \cdot \frac{1}{2} \cdot \left\{ \frac{1}{2\lambda} + \frac{1}{2\sqrt{2}\lambda} \sin \frac{\pi}{4} \right\} \\ &= \frac{1}{2} \frac{P}{a} \left[-\left(\frac{L}{2}\right)^5 - \frac{9}{32} L^5 \right]. \end{aligned}$$

$$\Rightarrow A = 226 \times 10^{-3}.$$

From Graph 6 it is seen that better agreement is obtained with this function, however some local errors still remain.

The photographs of contours of slopes are finally compared with contours of slopes obtained through Finite Element Method (see Chapter 3) and presented in Figure 11.

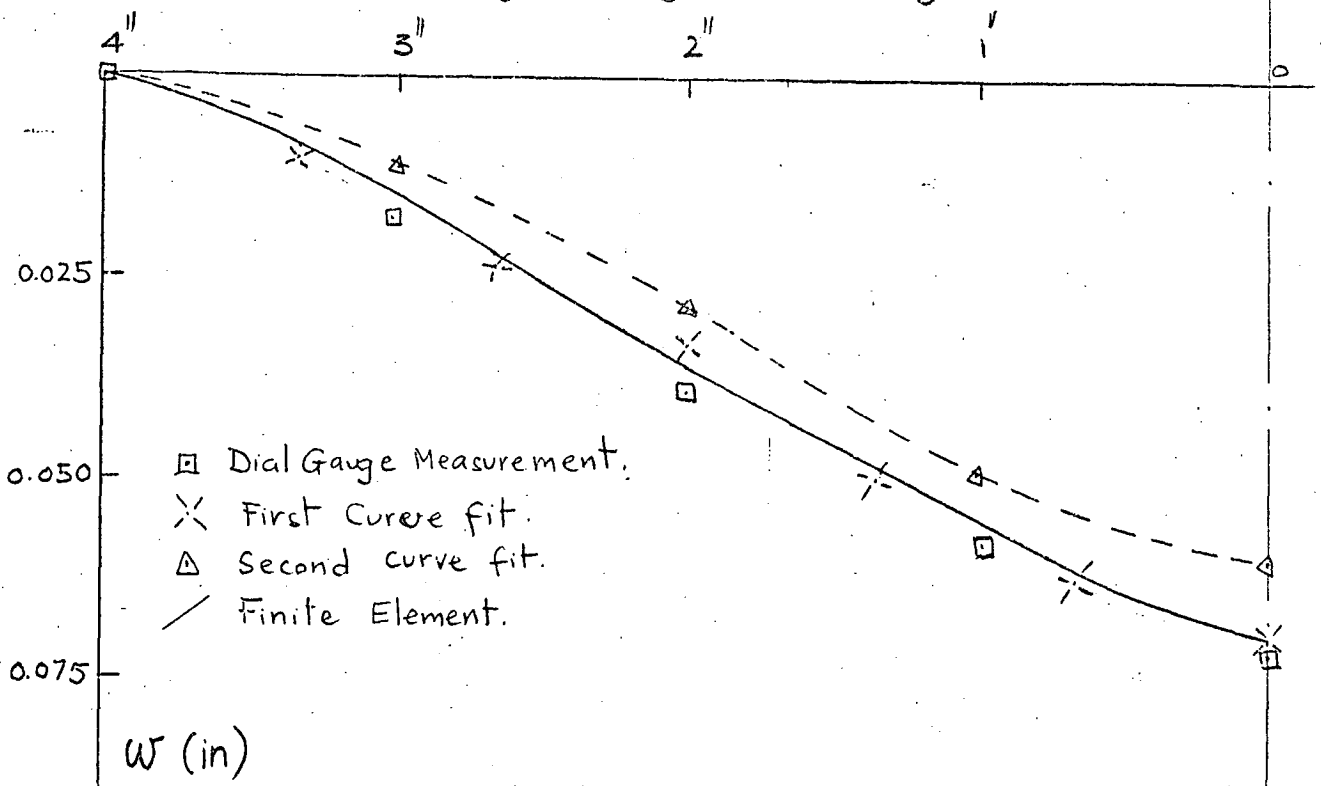
Variation along center line in transverse direction.



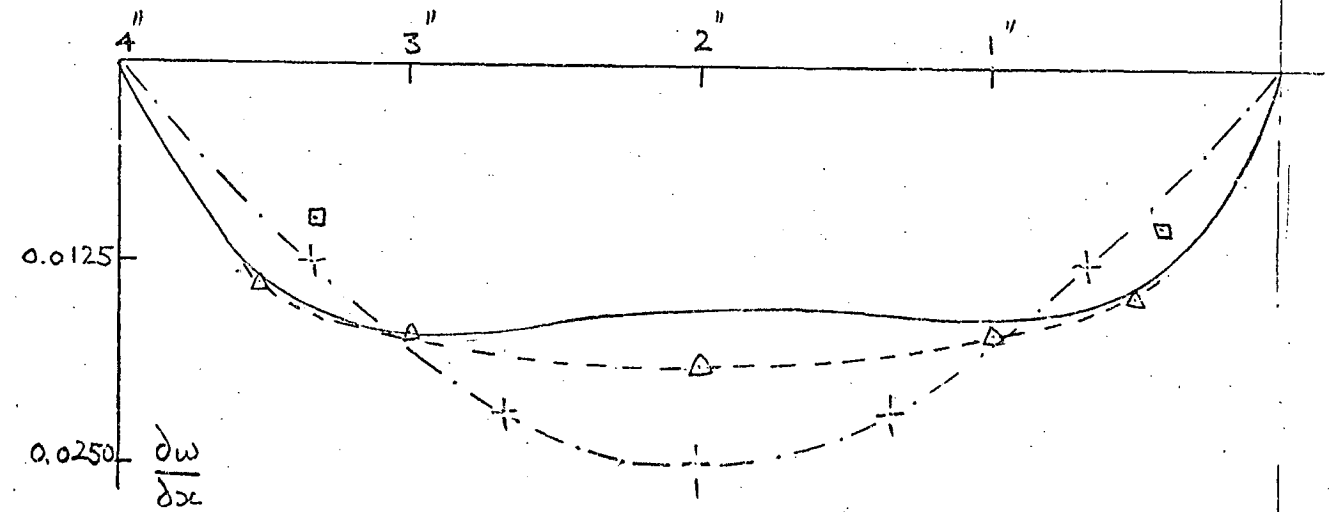
Graph 6(i)

Variation along a longitudinal edge.

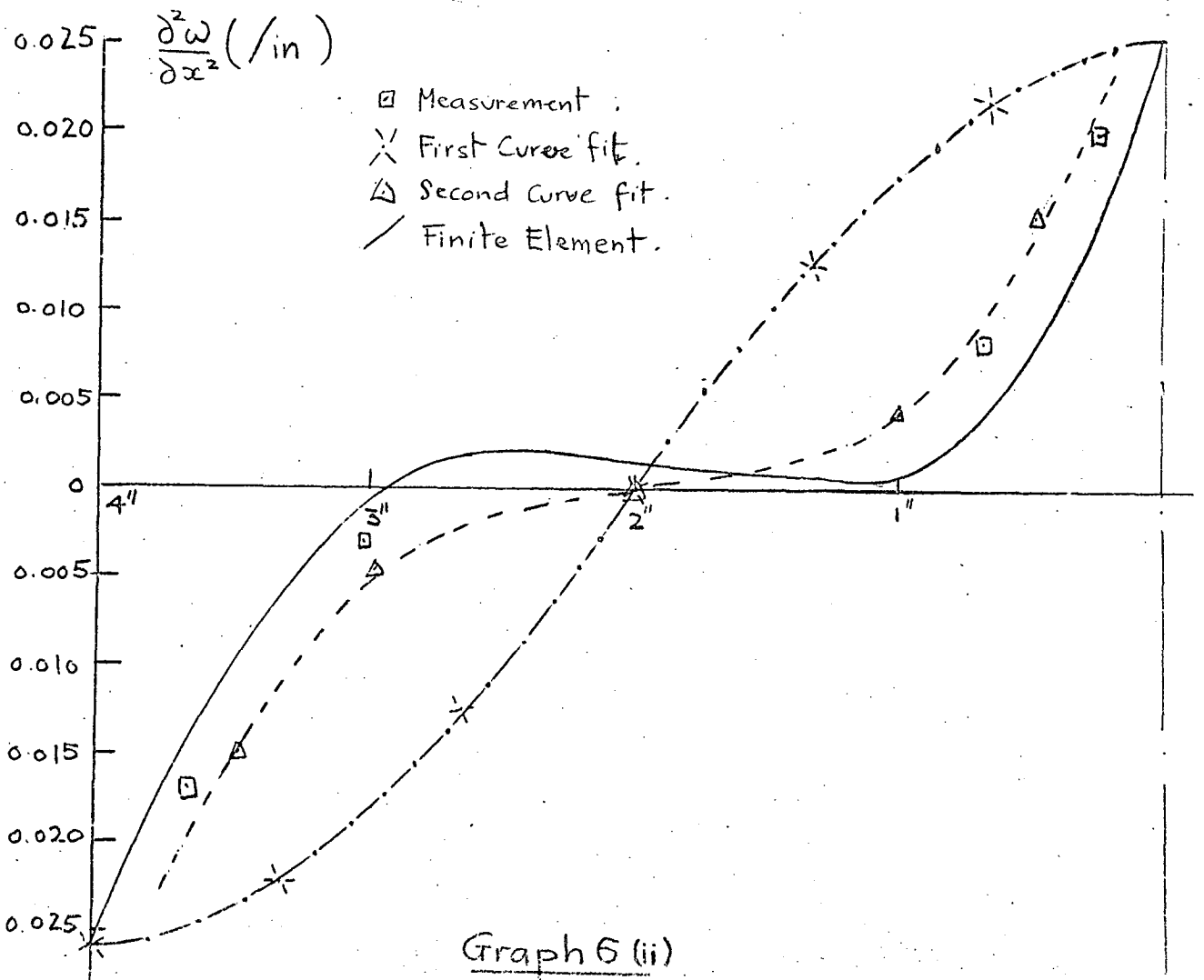
Deflection



Longitudinal Slope



Longitudinal Curvature



Graph 6 (ii)

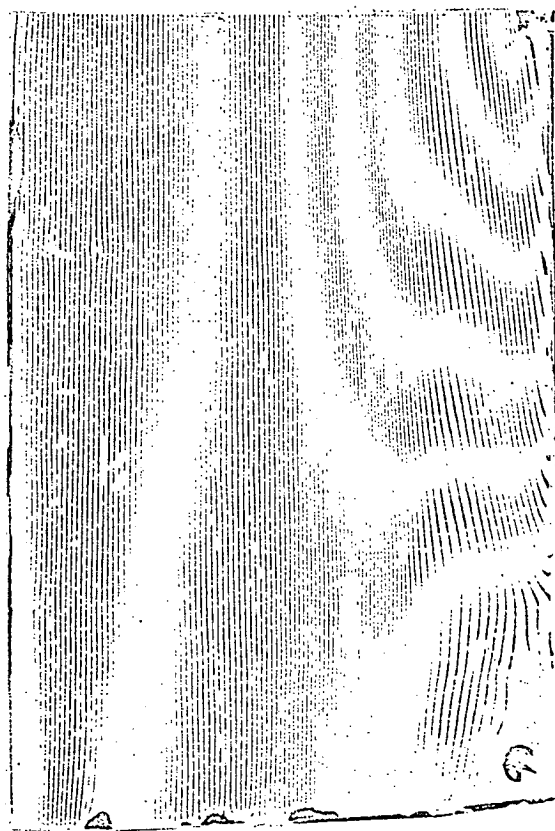
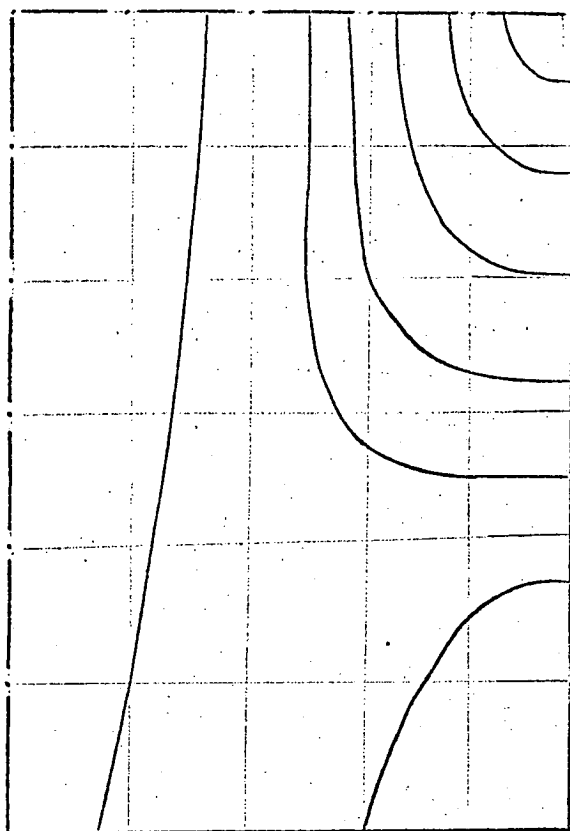
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CONTOURS OF SLOPES IN LONGITUDINAL AND TRANSVERSE DIRECTION
FOR A QUARTER OF THE SHELL.

TRANSVERSE SLOPE

FINITE ELEMENT

MEASUREMENT

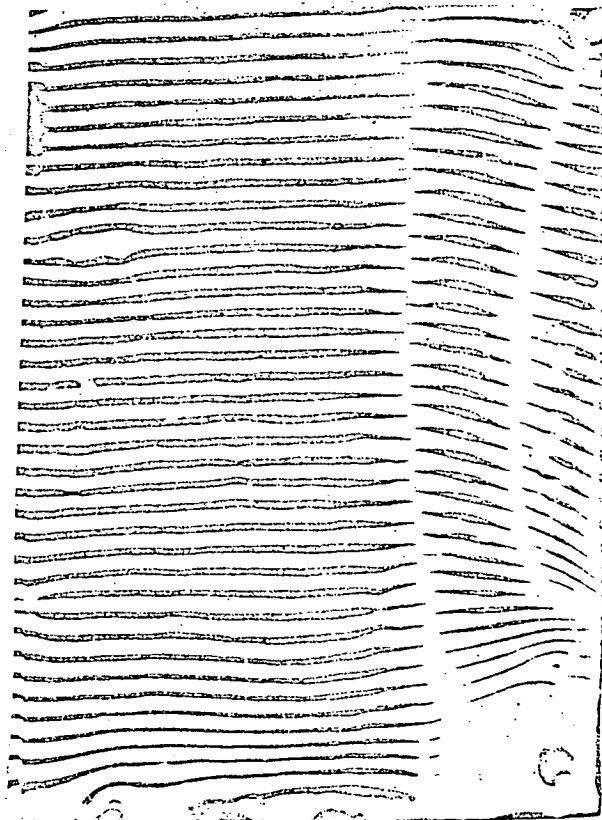
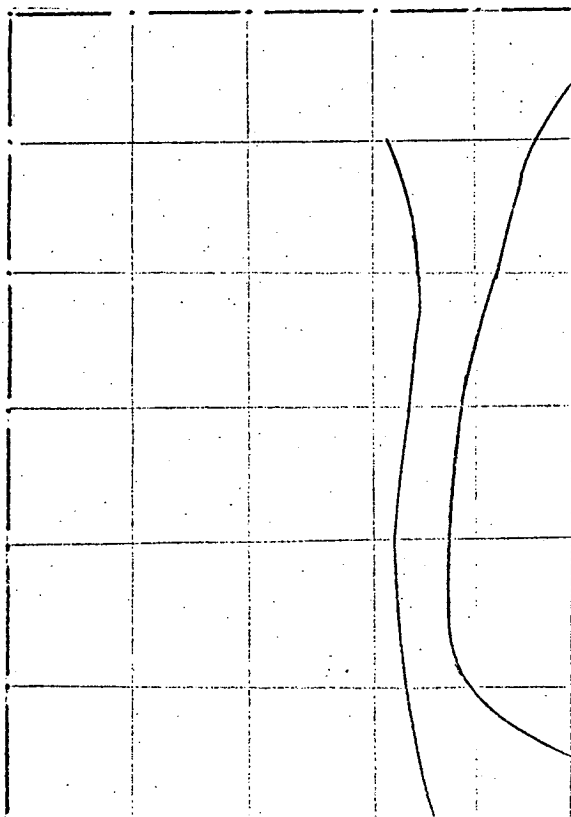


Increment of 0.0066

LONGITUDINAL SLOPE

FINITE ELEMENT

MEASUREMENT



Increment of 0.0066

FIGURE 11

2.8 Use of the method in shell experiment and design:

The experimental method presented above gives the means of recording of contours of slope in both longitudinal and transverse directions. The error of the measurement is within 5 - 10% limit provided that

- shell angle is less than 80°
- no large rigid body rotation and translation
- deflections are small, radial and tangential displacements may cause errors but they can be corrected.

The method is valuable because it gives an overall picture of the bending deformation of the whole shell surface. Even if the numerical results are not entirely accurate, the method still gives an indication of the functional form of the bending deformation of a shell and it identifies regions where bending is most dominant. This information provides a valuable guide for the shell designer.

The simplicity of the experimental technique is a great advantage of the method. However, the method only gives information on bending deformation. The membrane stresses of the shell must be obtained through another means such as strain gauge readings.

The wealth of information accumulated from the photograph offers many uses. Integration and differentiation of the slope values will give deflections and curvatures respectively. Direction of the contours can be used as a guide to the design of reinforcement for concrete shells (see Chapter IV). The method can also be used as an overall check on other methods of calculation (see Chapter III). The information also supplies a basis upon which simpler model for shell bending can be formulated as demonstrated in the above shell problem.

References

1. Ligtenberg, F.K. The Moire method - A new experimental method for the determinations of moments in small slab model. Experimental Stress Analysis Vol. XII No. 2, 1955.
2. Osgerberby, C. Application of the Moire Method for use with cylindrical surfaces. Experimental mechanics, July, 1967.
3. Crawford B.E Honours thesis - University of Tasmania.

CHAPTER III

FINITE ELEMENT METHOD IN SHELL ANALYSIS

3.1 Introduction:

The basic procedure of the finite element method used in structural analysis can be found in several texts (References 1, 2 & 3). Again, the three basic sets of data are the equations of statical equilibrium, the equations of geometry of deformation and the stress-strain relations. Originally, the method was formulated by assuming a displacement field; the principle of minimum potential energy was used to obtain a valid set of equations of statical equilibrium, i.e. the displacement method.

Alternatively, one can assume an equilibrating internal force field and use the complementary energy principle to determine the associated generalized displacements, i.e. the force method. Only the displacement method is used here and it is briefly summarized for completeness' sake.

(i) The structure is divided by imaginary lines into a number of regions. Each region is called an element. An element can be either one, two or three dimensional and can be of any arbitrary shape.

(ii) The elements are interconnected at a number of nodal points. The displacement of these points are taken as the basic parameters to define the deformed shape of the structure.

(iii) A function (usually polynomial) is chosen to define the state of displacement within the element. Strains and stresses are defined through this function and the elastic properties of the material. The function is known as the 'shape function'.

(iv) Loading on the structure is replaced by an equivalent set of generalized forces acting at the nodes.

The overall equilibrium of the system results in a relationship of the form

$$\{F\} = [K] \{\delta\}$$

Where $\{F\}$ is a column vector representing the forces acting at the nodes

$\{\delta\}$ is a column vector representing the corresponding modal displacements.

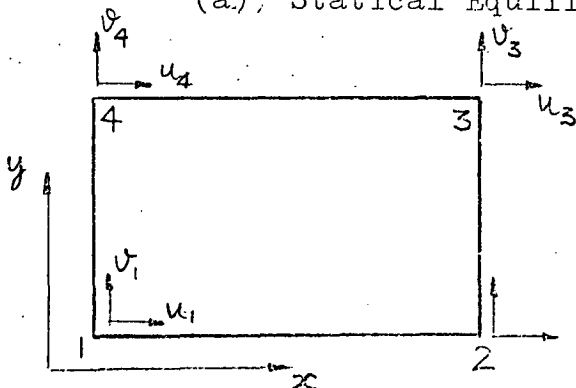
The matrix $[K]$ is known as the stiffness matrix of the structure and it is evaluated through the use of energy method.

The method as presented above falls within the framework of ^{the} Rayleigh-Ritz method; the object is to find a solution to the problem by using a guessed function for the deformed shape of the structure and the principle of minimum potential energy. The use of energy principle allows the equations of statical equilibrium to be satisfied on the average. Since the guessed pattern of deformation is used only over a small region, it can be intuitively seen that as the elements decrease in size, the solution will in general converge to the exact one. Certain convergence criterions have been established and can be found in references 1,2.

3.2 A study of the stiffness matrix:

The main step in the finite element method is to find the stiffness matrix for an element given an assumed deformation pattern; i.e. to find a set of forces necessary to produce the chosen deformation pattern. This can be done in several ways. The most convenient way is to restrict the necessary forces to the nodal forces. Once the notion of equivalent nodal forces is introduced, the question is how to relate them to the actual distributed forces; the energy methods are used for this purpose. To gain a better understanding of this method, the stiffness matrix for a rectangular element under conditions of plane stress is derived by both direct consideration of statical equilibrium and energy methods.

(a). Statical Equilibrium Consideration:



Consider a rectangular element a x b, having 4 nodes. In plane stress problem, each node has two degrees of freedom - the horizontal and vertical displacements - giving a total 8 degrees of freedom for the element.

The displacement vector $\{\delta\}$ is written as

$$\{\delta\} = \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}$$

The equivalent force vector is

$$\{F\} = \{U_1, V_1, U_2, V_2, U_3, V_3, U_4, V_4\}$$

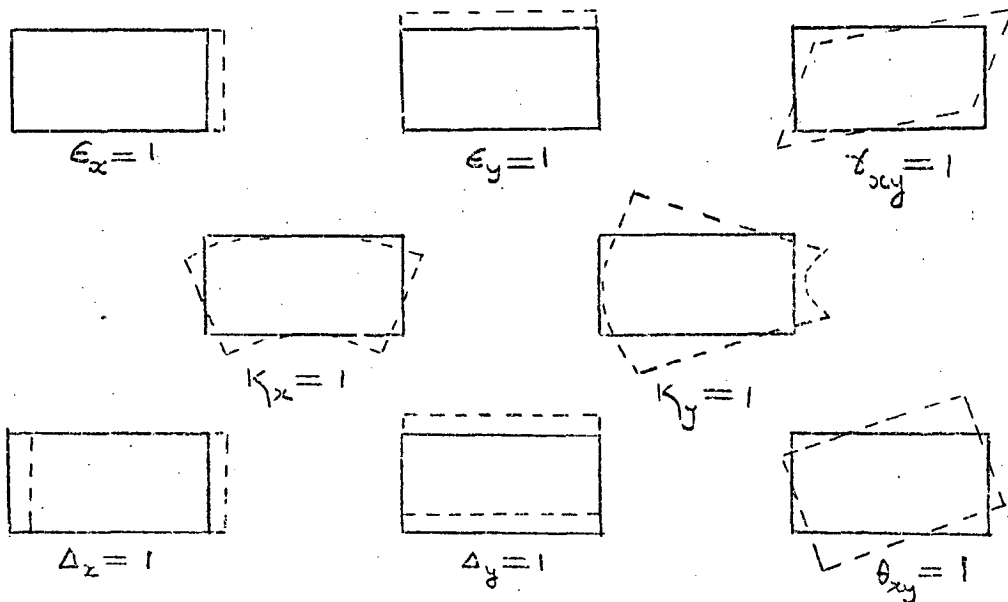
The stiffness matrix is defined as

$$\{F\} = [K] \{\delta\}$$

This can be found by proper combination of five basic stress conditions

$$\epsilon_x = 1 ; \quad \epsilon_y = 1 ; \quad \gamma_{xy} = 1 ; \quad \kappa_x = 1 ; \quad \kappa_y = 1 ;$$

and three rigid body movements: two translations in x and y direction and a rotation of the element about the normal axis of its plane. (Fig. 1)



Appendix 1 gives a detail derivation of the forces involved in each case and the resultant stiffness matrix. For later comparison of a particular component, the value for the force U_1

for $u_1 = 1, v_1 = u_2 = \dots = v_4 = 0$ is quoted here:

$$U_1 = \left\{ \frac{1}{4} \frac{b}{a} - \frac{1-\nu^2}{12} \frac{b}{a} - \frac{1}{8} (1-\nu) \frac{a}{b} \right\} \frac{Et}{1-\nu^2}$$

Various terms of this expression can be identified.

The term $\frac{1}{4} \frac{b}{a}$ comes from direct strain in x or y direction

The term $\frac{1-\nu^2}{12} \frac{b}{a}$ comes from bending deformation pattern

The term $\frac{1-\nu}{8} \frac{a}{b}$ comes from the shear consideration

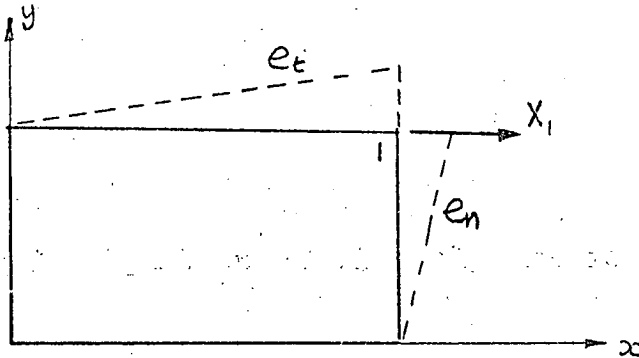
All five basic stress components are represented in this expression.

(b) Energy consideration:

Use of contour integration.

Consider eight nodal forces (X_1, Y_1, \dots)

Given the node 1 a displacement du_1 only the force X_1 does work.



$$X_1 du_1 = t \oint \sigma ds \frac{de_n}{du_1} du_1 + t \oint \tau ds \frac{de_t}{du_1} du_1$$

where σ = normal edge stress e_n = normal displacement

τ = tangential edge stress e_t = tangential displ.

For

$$u_1 = 1 \quad \frac{de_n}{du_1} = e_n^{u=1} \quad \frac{de_t}{du_1} = e_t^{u=1}$$

Then

$$U_1 = t \oint (\sigma e_n^{u=1} + \tau e_t^{u=1}) ds$$

The integration can only be carried out if the variations of displacements e_n, e_t are specified along the boundaries.

For example, if e_n, e_t vary linearly along the edge of the element, we get

$$U_1 = \int_0^b \frac{Et}{(1-\nu^2)} \frac{y}{ab} dy + \int_0^a \frac{Et}{2(1+\nu)} \frac{x}{ab} dx$$

$$= \frac{Et}{(1-\nu^2)} \left\{ \frac{1}{2} \frac{b}{a} + \frac{1}{4} (1-\nu) \frac{a}{b} \right\}$$

It is seen that the assumption of linear displacement variation excludes the bending deformation pattern.

(c) Use of Area Integration:

The contour integration can be replaced by an area integration. If this is done, the state of displacement within the element must be specified by a shape function.

The strain energy expression is:

$$U = \int_{\text{Area}} \left\{ \int_0^{\epsilon_x} \sigma_x d\epsilon_x + \int_0^{\epsilon_y} \sigma_y d\epsilon_y + \int_0^{\gamma_{xy}} \tau d\gamma_{xy} \right\} dx dy$$

$$= \sum_{i=1}^4 \int_0^{u_i} U_i du_i - \sum_{i=1}^4 \int_0^{v_i} V_i dv_i$$

The integrations can be performed subject to defining the variations of the quantities over the area of the element. Variation of U with respect to displacements at the nodes gives a set of equations of the type:

$$\frac{\partial U}{\partial u_i} = \sum k_{ij} u_j - U_i = 0$$

$$\frac{\partial U}{\partial v_i} = \sum k_{ij} v_j - V_i = 0$$

These equations can be identified as valid equations of statics of the element.

This set of equations can be rewritten as:

$$\{F\} = [k] \{\delta\}$$

where $\{\delta\}$ is the displacement vector.

$\{F\}$ is the corresponding force vector

and $[k]$ is the stiffness matrix.

This derivation of stiffness matrix is the standard method used in the Finite Element Approach.

For the particular case of rectangular plane stress element considered above, the stiffness matrix can be obtained by:

(i) Using a shape function with 8 coefficients³ such as

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

$$v = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy$$

results in stiffness coefficient of the type

$$U_i = \left[\frac{1}{3} \frac{b}{a} + \frac{1}{6} (1-\nu) \frac{a}{b} \right] \frac{Et}{1-\nu^2}$$

(ii) Using the combination of two triangular elements with shape functions

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$v = \alpha_4 + \alpha_5 x + \alpha_6 y$$

results in stiffness coefficient of type

$$U_i = \left[\frac{1}{2} \frac{b}{a} + \frac{1}{4} (1-\nu) \frac{a}{b} \right] \frac{Et}{1-\nu^2}$$

It is seen again that the bending component $(1-\nu^2)\frac{a}{b}$ does not appear in both cases. The use of the shape function $u = \alpha_1 + \alpha_2 x + \alpha_3 y$ gives the same stiffness coefficients as those obtained by assuming a linear variation of displacement along the boundaries and using contour integration. In fact the two processes are identical throughout.

Despite various numerical values obtained for the direct stress and shear terms, the ratio of the terms is always the same and equal to 2.

Conclusion - The standard method of using shape functions and the principle of minimum potential energy is very efficient in obtaining the stiffness matrix. The use of an energy method, however, tends to obscure the equilibrium principles and it hides the weakness and deficiency of the shape functions used, as demonstrated above. The use of shape functions in general is an approximation which does not necessarily include all the basic patterns of deformation. Thus the accuracy of the method largely depends on the type of function used. It is hoped that with the use of a large number of elements, the complicated deformation of the structure can be adequately represented by the combination of a few basic modes of deformation. There are two possible approaches to the problem:-

To use simple shape functions (fewer degrees of freedom per element) and a large number of elements.

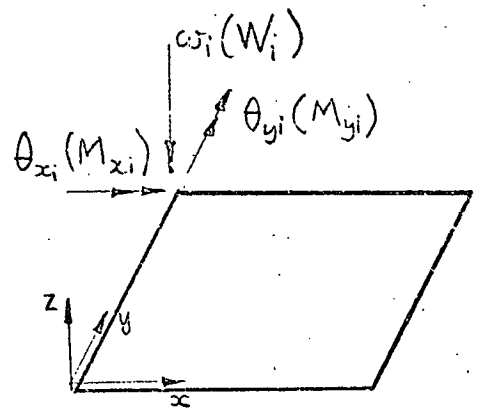
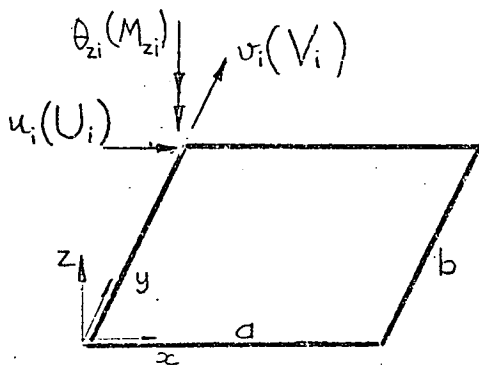
To use complicated shape functions (more degrees of freedom per element), the element is now more adaptable to many kinds of deformations therefore fewer elements are needed in general.

The important point is the total number of equations ultimately to be solved. No satisfactory agreement has been reached as to which approach is better.

3.3 Finite Element Analysis of Cylindrical Shells:

(a) Shell as an assembly of flat plate elements⁴:

In the analysis of flat plates by the Finite Element method, the plate is divided into a number of elements by imaginary lines and a shape function is used to define the state of deformation within the element. Thus, there is an approximation in the deformed shape but there is no approximation in the sub-division of the undeformed state of the plate. In the analysis of shells a further approximation is often made. It is assumed that the behaviour of a cylindrical shell can be represented by a surface built up of flat elements. This assumption is based on the well-known similarity between the behaviour of cylindrical shells and folded plates. Intuitively one feels that convergence to the curved shell must occur as the size of the element decreases.



Take a co-ordinate system as follows x,y axes in the plane of the element z-axis normal to the plane. Each node of the element has in general six degrees of freedom, three displacements and three rotations about the axis. The displacement vector for each node is

$$\{\delta_i\} = \{u_i, v_i, w_i, \theta_{xi}, \theta_{yi}, \theta_{zi}\}$$

Corresponding to these degrees of freedom are the six generalized forces

$$\{F\} = \{U_i, V_i, W_i, M_{xi}, M_{yi}, M_{zi}\}$$

In order to use the existing models of plane stress and bending of plates, two further assumptions must be made:

(i) Rotation θ_z about the Z-Axis is neglected. This is a geometrical restriction on the possible deformation of the element. Physically it means that the plate element is infinitely stiff to a rotation about an axis perpendicular to its own plane.

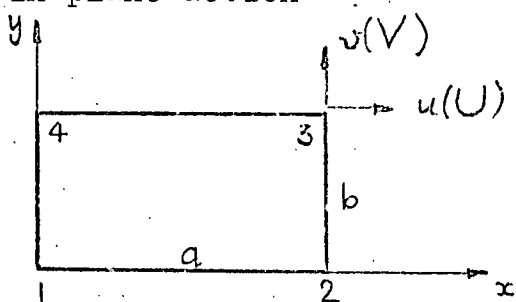
(ii) There is no interaction between in-plane stresses and bending stresses at element level. This is a statical restriction on the stress distribution within the element. For the flat element, it implies that the deformation in bending must be small so that no significant membrane actions are introduced. For the actual curved shell element this is not true.

Analysis in Chapter I showed that the interaction does exist. However, the stress-strain relation used in subsequent simplified standard analytical methods also neglect the interaction components.

(b) The stiffness matrix of an element:

For the particular case of cylindrical shell, the structure can be well represented by rectangular element. Subject to the above assumptions, the in-plane and bending action for each element can be dealt with separately.

In-plane action



Each node has two degrees of freedom. The deformation vector of the node i is

$$\{\delta_p^i\} = \{u_i, v_i\}$$

The corresponding force vector is.

$$\{F_p^i\} = \{U_i, V_i\}$$

The two vectors are related by the in-plane stiffness matrix $[k_p]$

$$\{F_p^i\} = [k_p] \{\delta_p^i\}$$

The shape function used for plane stress analysis are:

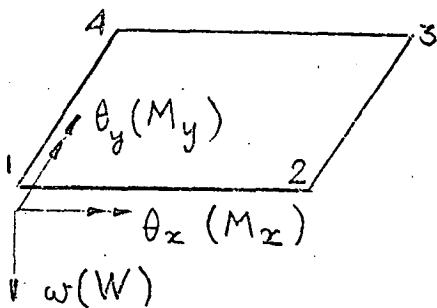
$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

$$v = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy$$

These shape functions satisfy the requirement of continuity of the in-plane displacements between adjacent elements. (Each component of displacement varies linearly along the sides of the element, the coincidence of displacement at the nodes therefore ensures the continuity of displacements of all intermediate points).

The first three terms of the shape functions give constant strain, the xy term introduces linear variation of a particular kind.

Bending action - Each node has three degrees of freedom.



The deformation vector is

$$\{\delta_b^i\} = \{w_i, \theta_{xi}, \theta_{yi}\}$$

The corresponding force vector is

$$\{F_b^i\} = \{W_i, M_{xi}, M_{yi}\}$$

The two vectors are related by the bending stiffness matrix $[k_b]$

$$\{F_b^i\} = [k_b] \{\delta_b^i\}$$

The shape function used for bending analysis is

$$w = \alpha_9 + \alpha_{10}x + \alpha_{11}y + \alpha_{12}x^2 + \alpha_{13}y^2 + \alpha_{14}xy + \alpha_{15}x^3 + \alpha_{16}x^2y + \alpha_{17}xy^2 + \alpha_{18}y^3 + \alpha_{19}x^3y + \alpha_{20}xy^3$$

This shape function is a complete cubic with two additional fourth order terms. Along a line $x = \text{constant}$ or $y = \text{constant}$, the deflection varies as a cubic. Along the edge of the element, this cubic is completely defined by four parameters (slopes and deflections at the two nodes). Thus continuity of deflection and slope along the edge is achieved. The slope normal to the edge varies parabolically and is not uniquely defined since only two slope values at the nodes are specified. Therefore in general a discontinuity in normal slope will occur. The curvatures vary linearly along lines $x=\text{constant}$ or $y=\text{constant}$, while the twist varies parabolically along these lines.

Combined Action: Combination of in-plane and bending deformations gives each node five degrees of freedom. The total degrees of freedom of the element (20) is the same as the total number of parameters required to define the three shape functions.

The displacement vector is $\{\delta^i\} = \{u_i, v_i, w_i, \theta_{xi}, \theta_{yi}\}$

The corresponding force vector is $\{F^i\} = \{U_i, V_i, W_i, M_{xi}, M_{yi}\}$

$$\{F^i\} = [k^i] \{\delta^i\}$$

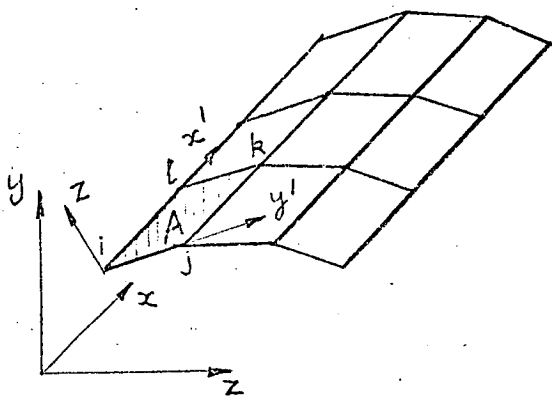
where

$$[k] = \begin{bmatrix} k_p & 0 \\ 0 & k_b \end{bmatrix}$$

$[k_p]$ and $[k_b]$ are respectively the in-plane and bending stiffness matrices. The zero submatrices indicate that there is no interaction between the two patterns of deformation at element level.

(c) Transformation of co-ordinates:

The stiffness matrix derived above used a local co-ordinate system. To assemble the elements it is necessary to transform the matrix to a common global co-ordinate system: x, y, z .



The local co-ordinate system of the element A is x', y', z' . The displacement vectors in local and global system are respectively $\{\delta_i'\}$ and $\{\delta_i\}$

They are related as follows

$$\begin{bmatrix} u_i' \\ v_i' \\ w_i' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}$$

$$\begin{bmatrix} \theta_{xi}' \\ \theta_{yi}' \\ \theta_{zi}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{bmatrix}$$

The three by three matrix in these relations is called the transformation matrix. It is seen to be made up of the cosines of the angles formed between the two set of axis.

The geometrical restriction $\theta'_{zi} = 0$ implies

$$-\sin \alpha \cdot \theta_{yi} + \cos \alpha \cdot \theta_{zi} = \theta'_{xi} = 0$$

$$\theta_{zi} = \theta_{yi} \tan \alpha$$

$$\begin{bmatrix} \theta'_{xi} \\ \theta'_{yi} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \cdot \tan \alpha \\ 0 & -\sin \alpha & \cos \alpha \cdot \tan \alpha \end{bmatrix} \begin{bmatrix} \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{bmatrix}$$

or

$$\begin{bmatrix} \theta'_{xi} \\ \theta'_{yi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\cos \alpha} \end{bmatrix} \begin{bmatrix} \theta_{xi} \\ \theta_{yi} \end{bmatrix}$$

The total transformation matrix thus becomes

$$\{\delta'_i\} = [T] \{\delta_i\}$$

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha & 0 & 0 \\ 0 & -\sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\cos \alpha} \end{bmatrix}$$

The stiffness matrix in global co-ordinate is $[k] = [T]^T [k'] [T]$

where $[T]^T$ is the transpose of $[T]$.

Note that this transformation procedure breaks down if the element is in the x-z plane, for then $\alpha = 90^\circ$, $\cos \alpha = 0$.

and $1/\cos \alpha = \infty$. Thus one has to choose the global co-ordinate system so that no element is in the x-z plane.

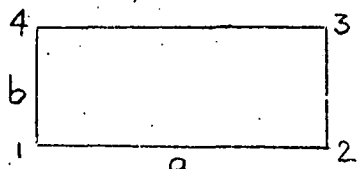
Solution for stresses and moments.

The assembly of all the elements and the application of loading results in a set of equations

$$[K] \{\delta\} = \{F\}$$

where $[K]$ is called the total stiffness matrix of the structure, $\{F\}$ is the loading vector.

This set of equations is solved for the displacement vector $\{\delta\}$. When the displacements of all the nodes are known, the internal stresses and moments can be obtained by back substitution. In general a node is the intersection of four elements, there are four sets of forces at each node from the four elements and in general they are not equal. There are many ways of averaging these forces, one of the most convenient way is to average the force for each element and assign the value to the centroid of the element.



$$\begin{aligned} \epsilon_x &= \frac{u_4 - u_1 + u_3 - u_2}{2a} ; \\ \epsilon_\phi &= \frac{v_2 - v_1 + v_3 - v_4}{2b} ; \\ \gamma_{x\phi} &= \frac{u_2 - u_1 + u_3 - u_4}{2b} + \frac{v_4 - v_1 + v_3 - v_2}{2a} ; \\ K_x &= \frac{\theta_{y1} + \theta_{y2} - \theta_{y3} - \theta_{y4}}{2a} ; K_\phi = \frac{-\theta_{x1} + \theta_{x2} + \theta_{x3} - \theta_{x4}}{2b} ; \\ K_{x\phi} &= \frac{-2(\omega_1 + \omega_2 - \omega_3 - \omega_4)}{ab} + \frac{-\theta_{y1} + \theta_{y2} + \theta_{y3} - \theta_{y4}}{4b} + \frac{\theta_{x1} + \theta_{x2} - \theta_{x3} - \theta_{x4}}{4a} \end{aligned}$$

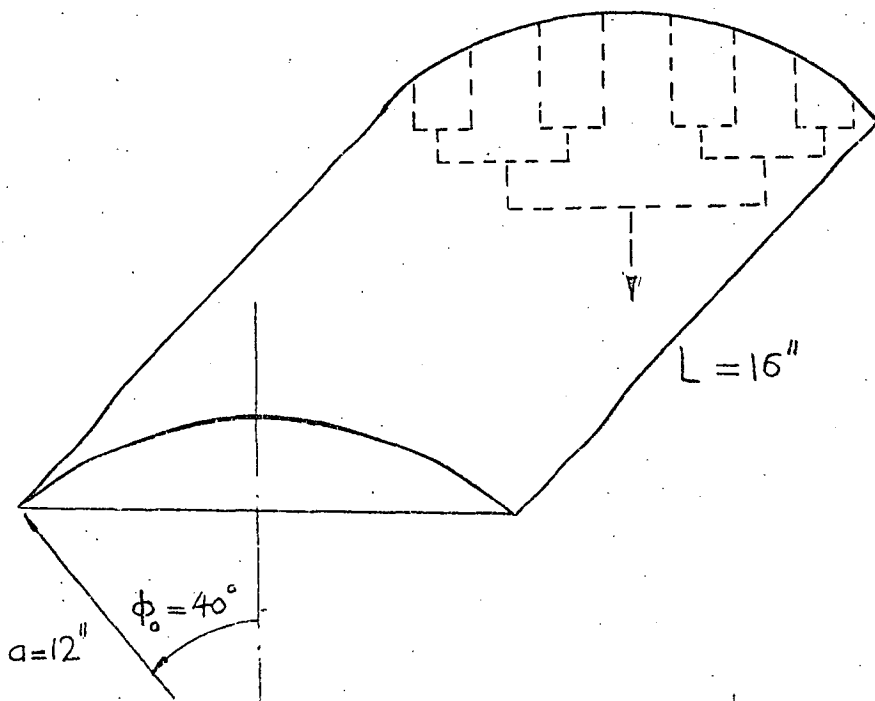
From these generalized strains, the stress resultants are obtained through the stress-strain relations.

$$\begin{aligned} N_x &= Eh (\epsilon_x + \nu \epsilon_\phi) ; M_x = Dh (K_x + \nu K_\phi) ; \\ N_\phi &= Eh (\epsilon_\phi + \nu \epsilon_x) ; M_\phi = Dh (K_\phi + \nu K_x) ; \\ N_{x\phi} &= \frac{Eh}{2(1+\nu)} \gamma_{x\phi} ; M_{x\phi} = Dh K_{x\phi} (1-\nu) . \end{aligned}$$

3.4 Analysis of a simply supported shell roof:

A shell roof supported by end traverses and free edges is analysed by various methods for comparison.

Shell dimensions and properties:



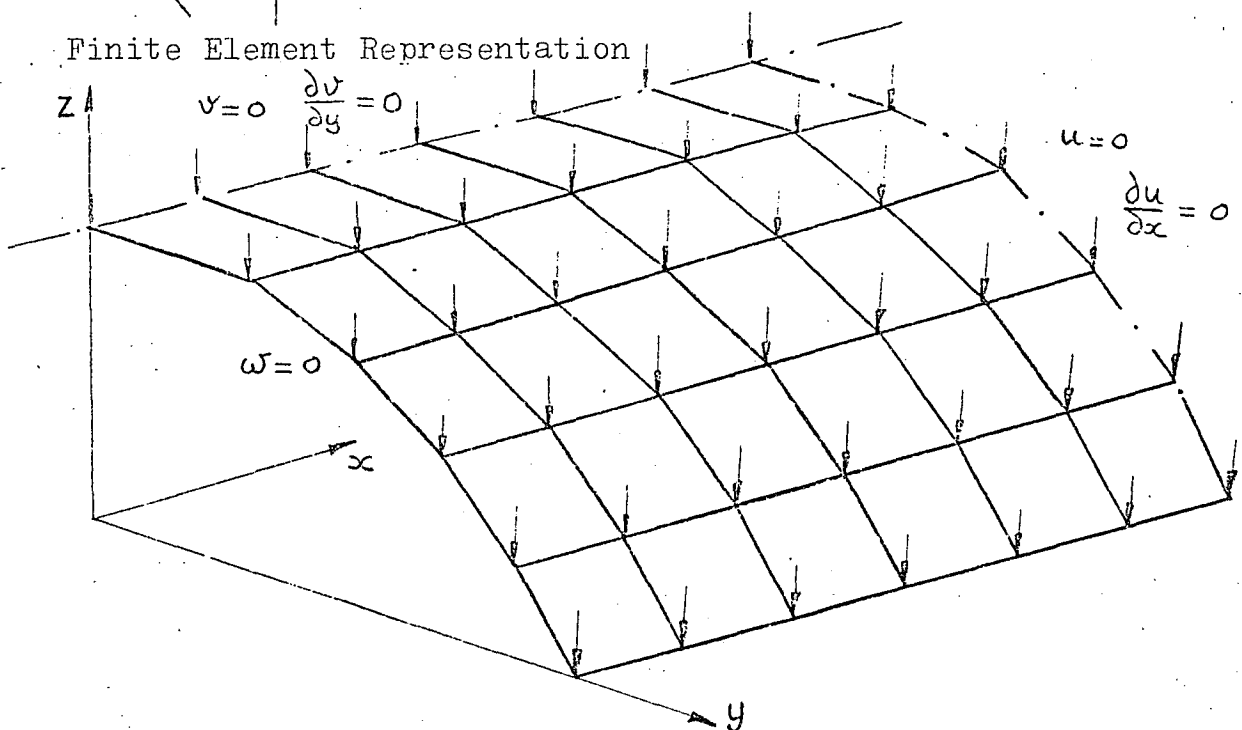
Steel shell model
with steel end
traverses

$$E = 30 \times 10^6 \text{ psi.}$$

$$h = 0.030 \text{ in.}$$

$$\nu = 0.33.$$

Loading is vertical
and uniformly distributed

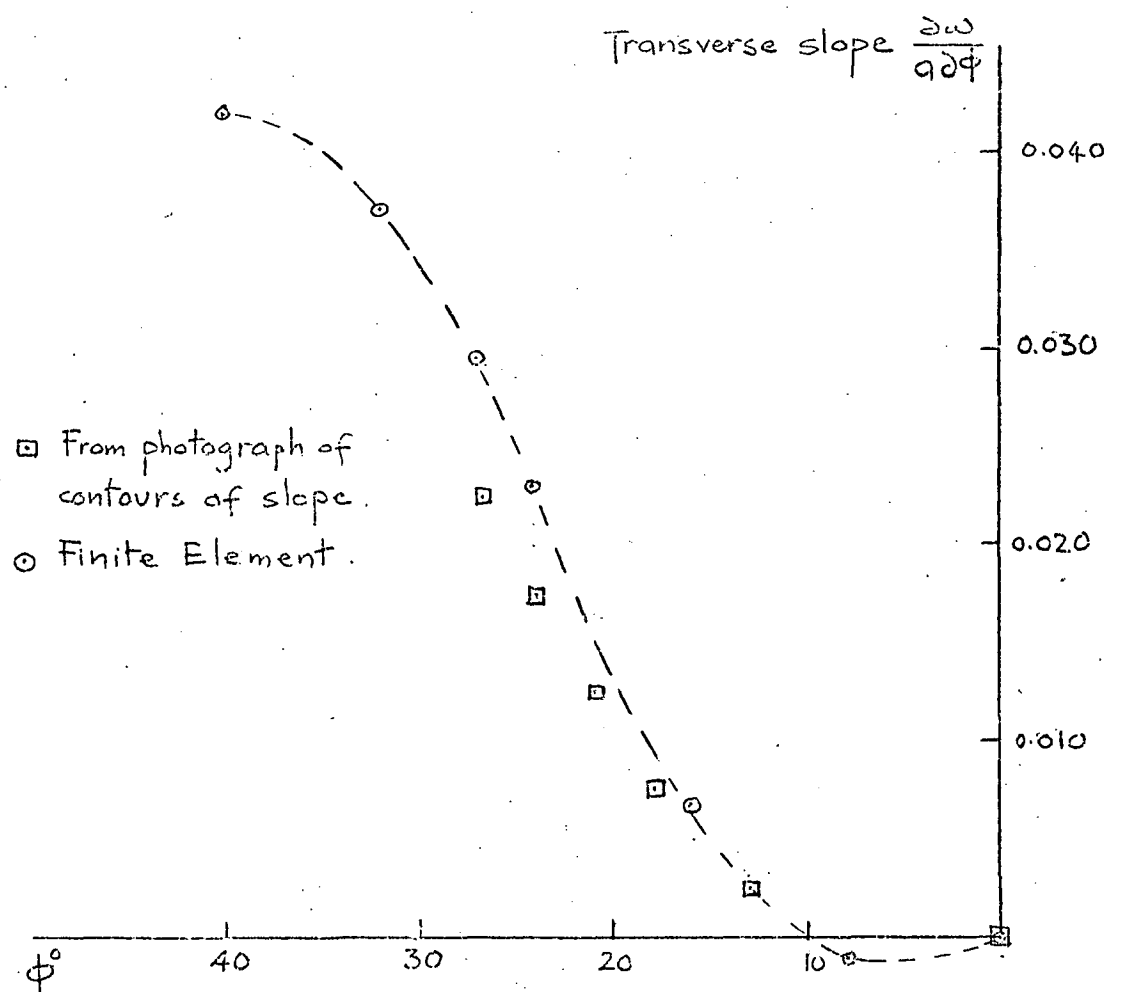
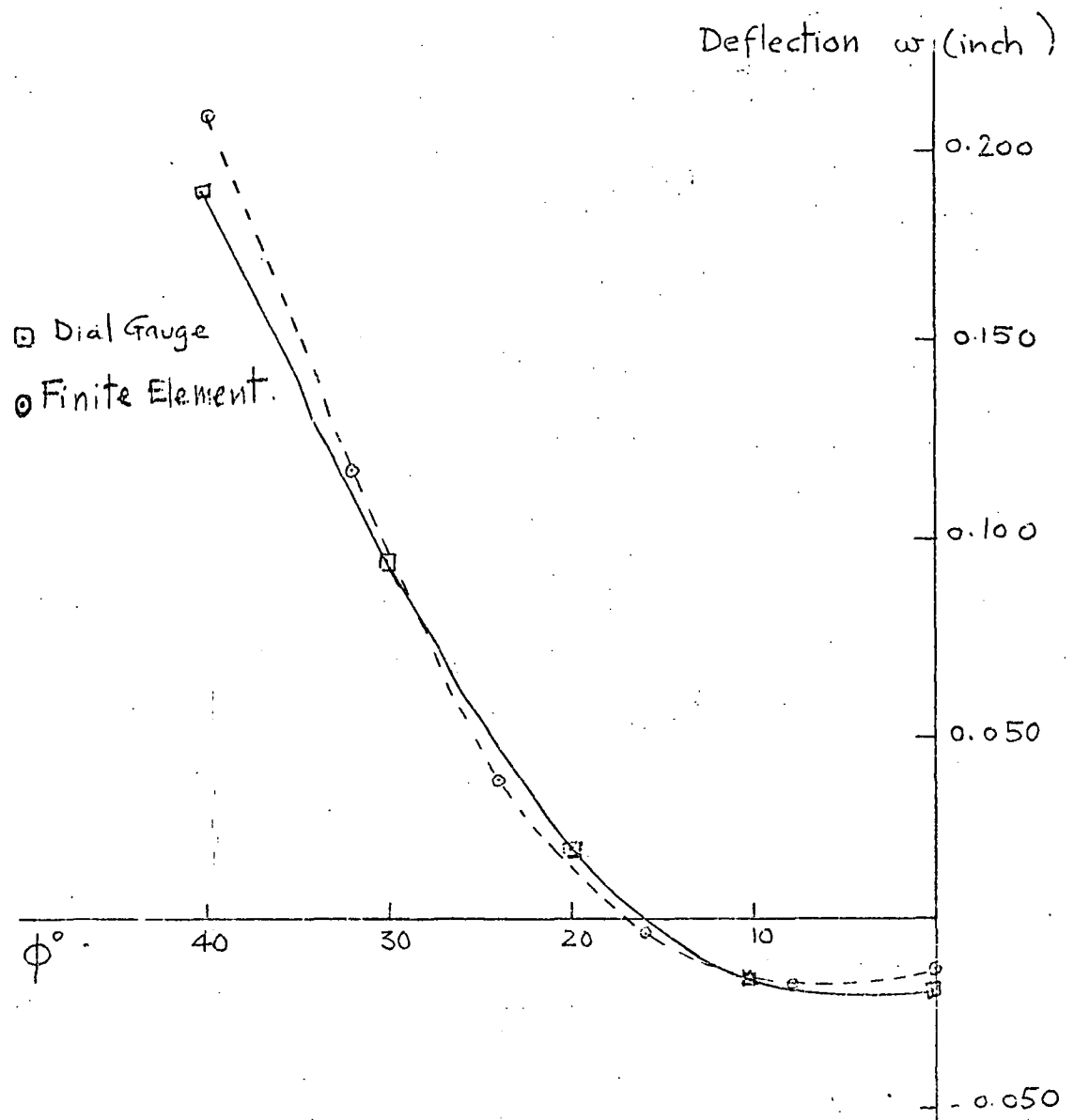


Only a quarter of the shell needs to be analysed due to its double symmetry.

Boundary conditions are:

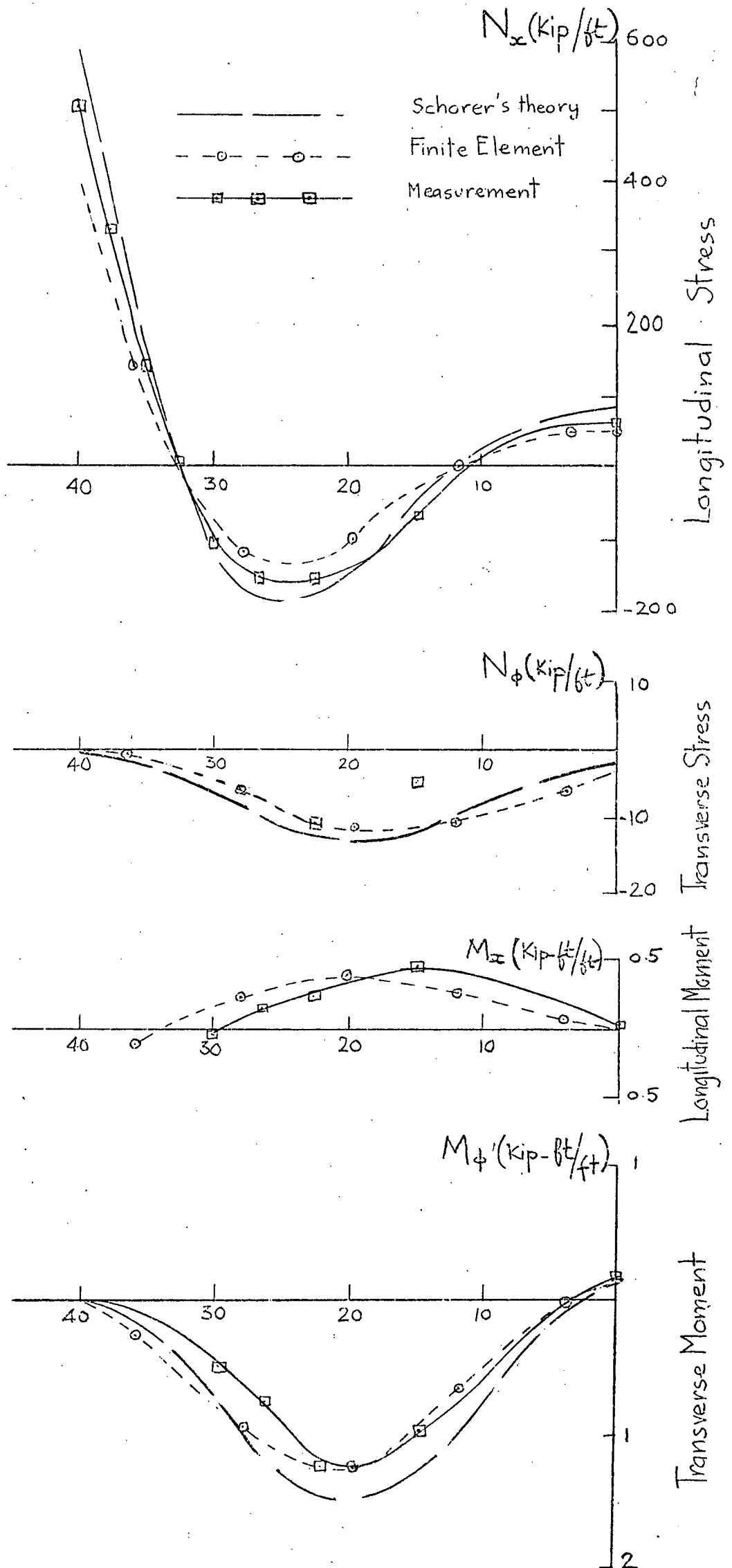
- (i) Along traverses $w = 0$, $v = 0$ (the traverses are assumed to be rigid in its own plane but flexible in the direction normal to it.
- (ii) Along centre line in x direction $v = 0$, $\frac{\partial w}{\partial y} = 0$.

Comparision of deformation



Graph 3.1 (a)

Comparison of stresses and moments.

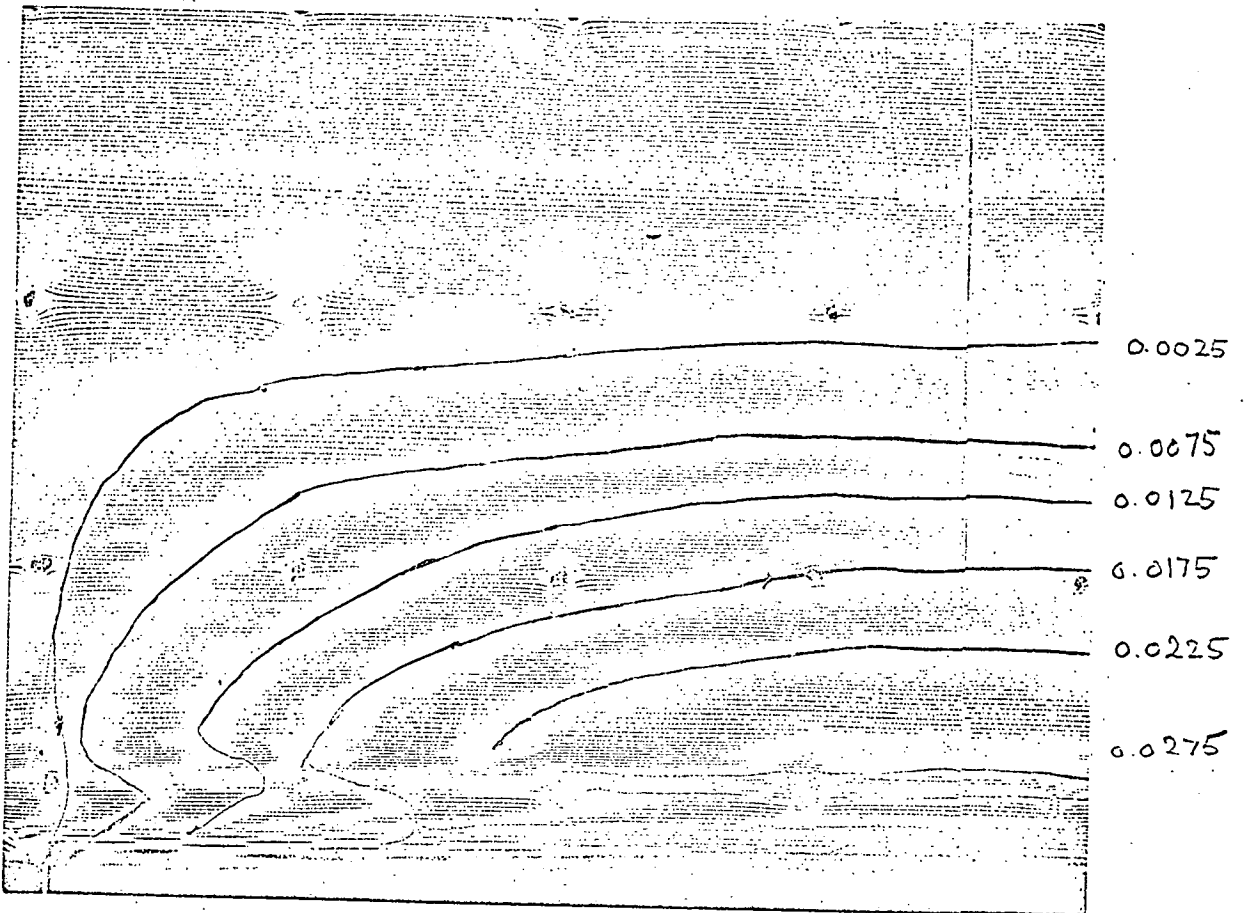


Graph 3.1(b)

CONTOURS OF TRANSVERSE SLOPE FOR A QUARTER OF A SHELL

(Vertical uniform loading)

MEASUREMENT



FINITE ELEMENT

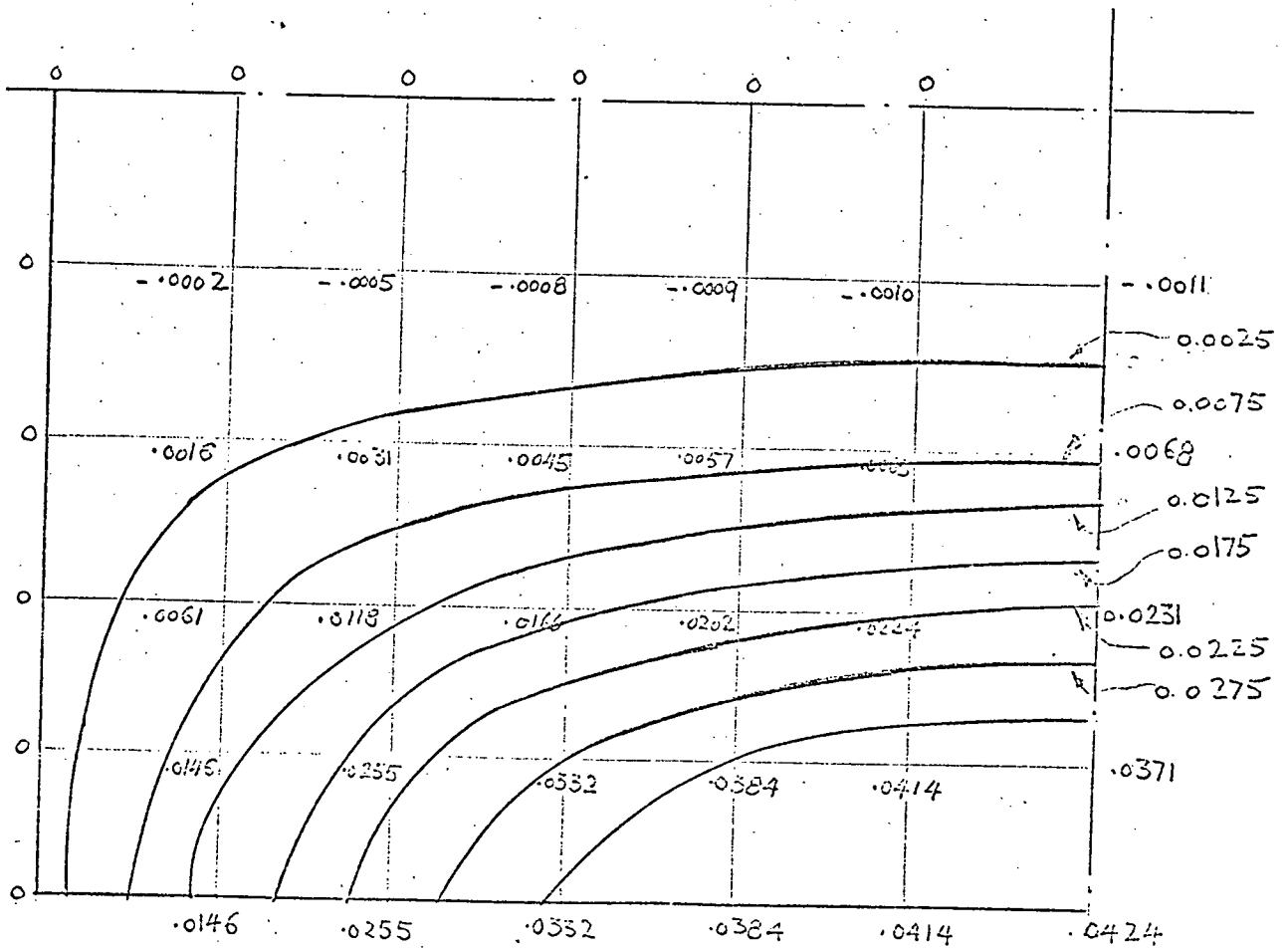


FIGURE 3.2 (a)

(iii) Along centreline in ϕ direction $u = 0$; $\frac{\partial w}{\partial x} = 0$

The shell is approximated by 30 flat plate elements, 6 in the longitudinal direction and 5 in transverse direction. There are 42 nodes with a total of 210 displacement parameters.

Comparisons with results from other methods and experimental measurements:

(a) Comparison of deformation:

Deformations of the shell was recorded by the Moiré - Lightenberg method described in Chapter II - Deflections were also measured with dial gauges as a check. Deformation was also obtained by analysis using Schorer's equation.

Results are presented in Figures 3.2 and Graph 3.1(a)

(b) Comparison of stresses and moments:

Values of stresses and moments from the Finite Element method are compared with measurements taken with strain gauges and results given by Schorer's theory.

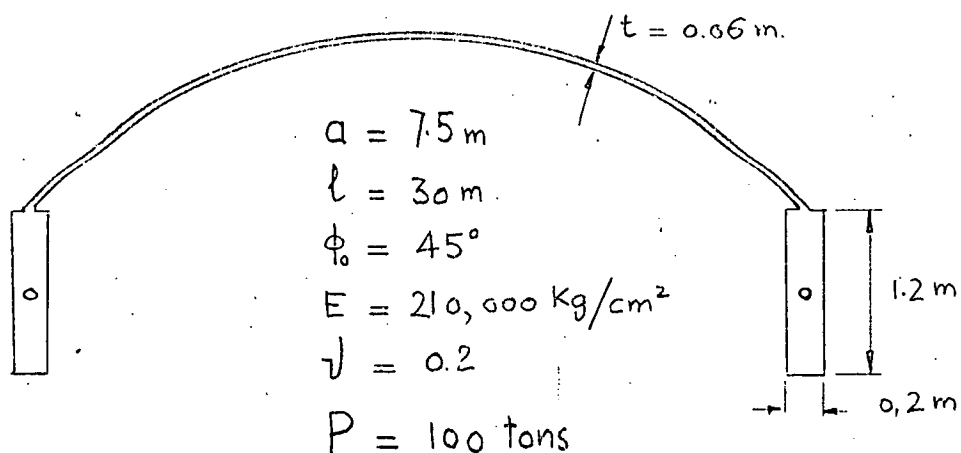
Results are plotted on Graph 3.1(b) for comparison.

Fairly good agreement exists between various methods for both displacements and stresses except in regions near the boundary where the variation is large. The finite element mesh was too coarse for a critical comparison in these regions.

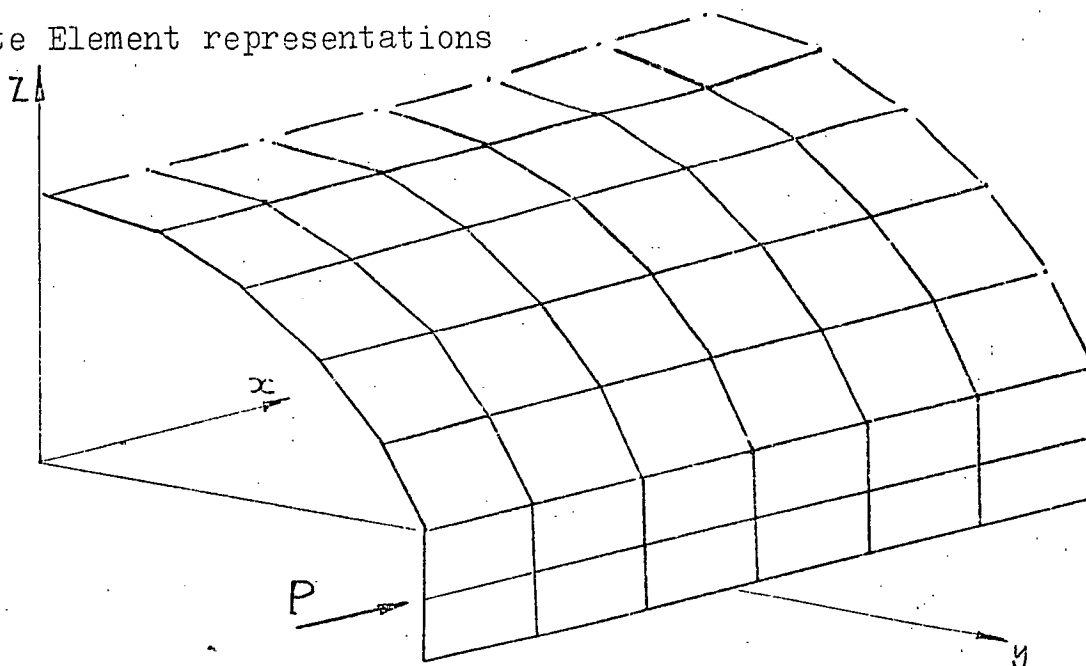
3.5 Analysis of Cylindrical Shell Roof with Prestressed Edge Beam:

One of the advantages of the Finite Element method as used for shell is the possibility of incorporating other structural members such as edge beams and traverses into the main programme for the shell. The interaction between various structural components can then be studied without making any further assumption about the stress distribution as often was done in other analytical method.

The problem of the prestressed edge beam can be solved by treating the edge beam as another element of the shell. The following shell with prestressed edge beam is solved by Finite Element method and compared with other solutions.



(a) Finite Element representations



This shell has been solved by Girkmann (Flachentragwerke) for the dead loading and by Dabrowsky for prestressing. Only a quarter of a shell needs to be analysed due to its double symmetry. The shell is approximated by 30 flat elements as in the previous section. The beam is approximated by two rows of vertical elements. Each row has six elements. The prestressing force acts at the centre of the beam section.

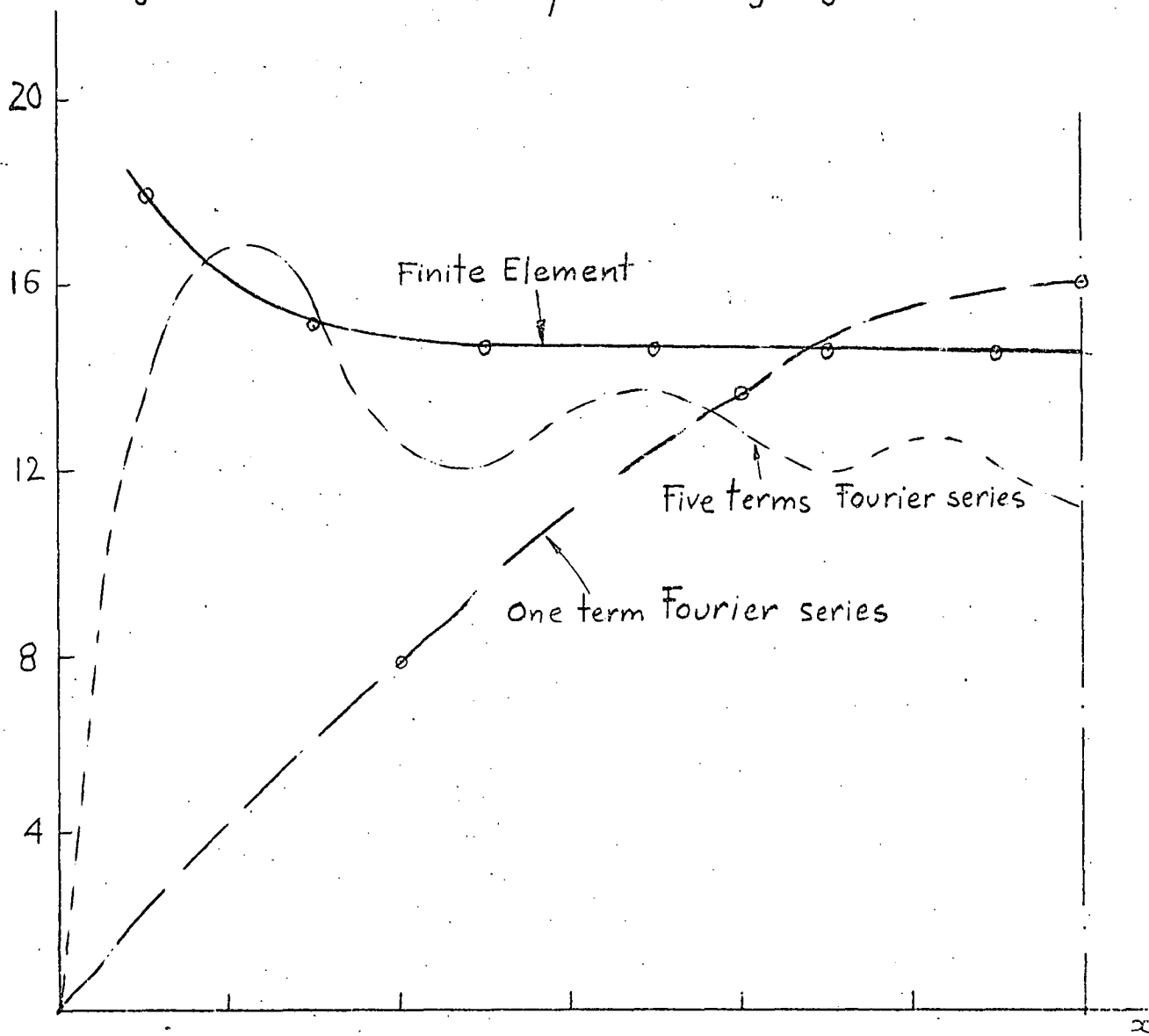
(b) Comparison with other analytical solution using Schorer's theory:

The standard way of handling of the prestressing load has been well presented by Gibson and Dabrowski. The prestressing force is usually represented by a Fourier series

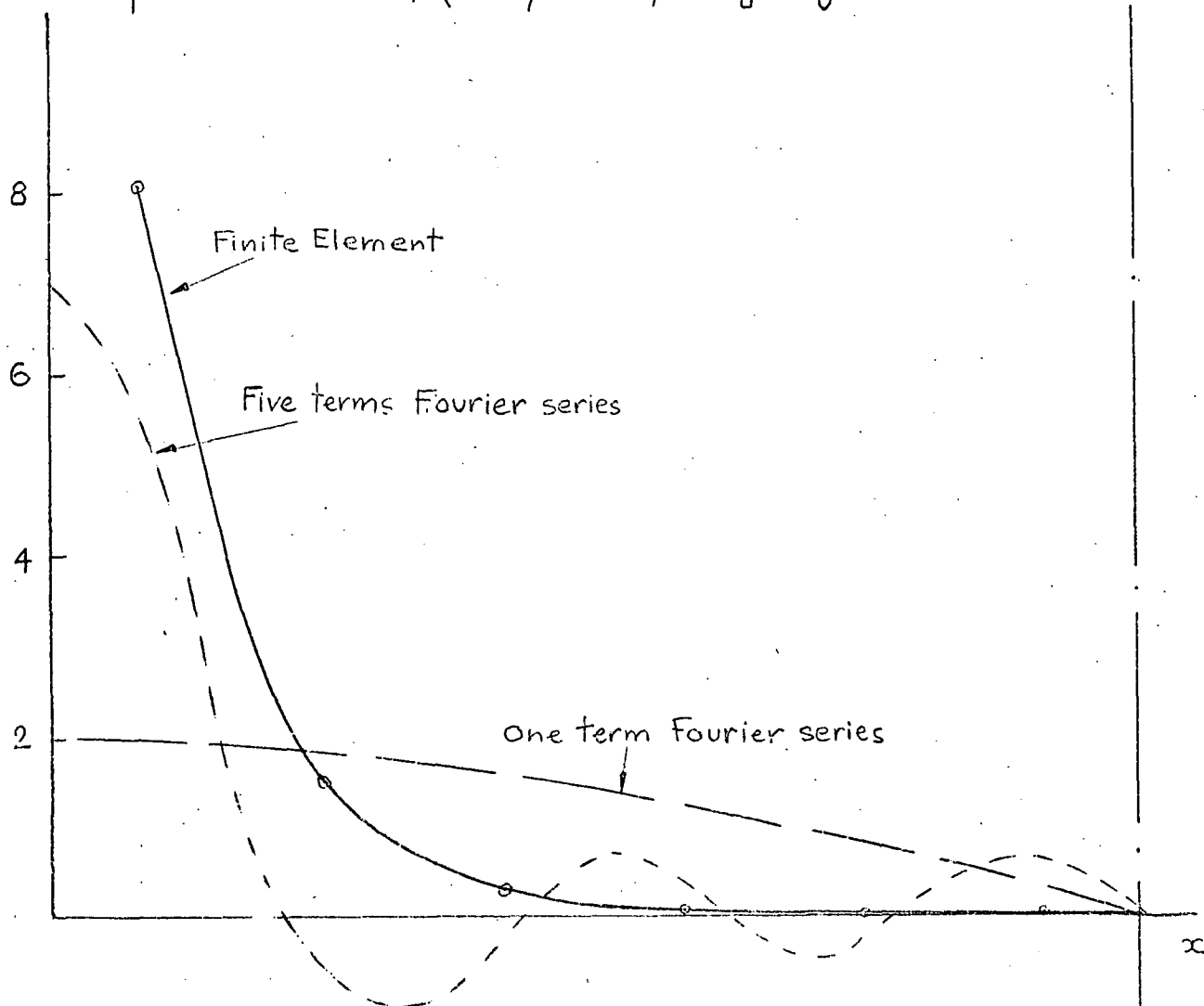
$$\frac{4}{\pi} P \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cos \frac{n\pi x}{L}$$

Gibson used only one term of the series. Dabrowsky showed that this was not adequate; he proposed to use at least five terms of the series to cover the increase in bending moments and shears at the outer part of the shell.

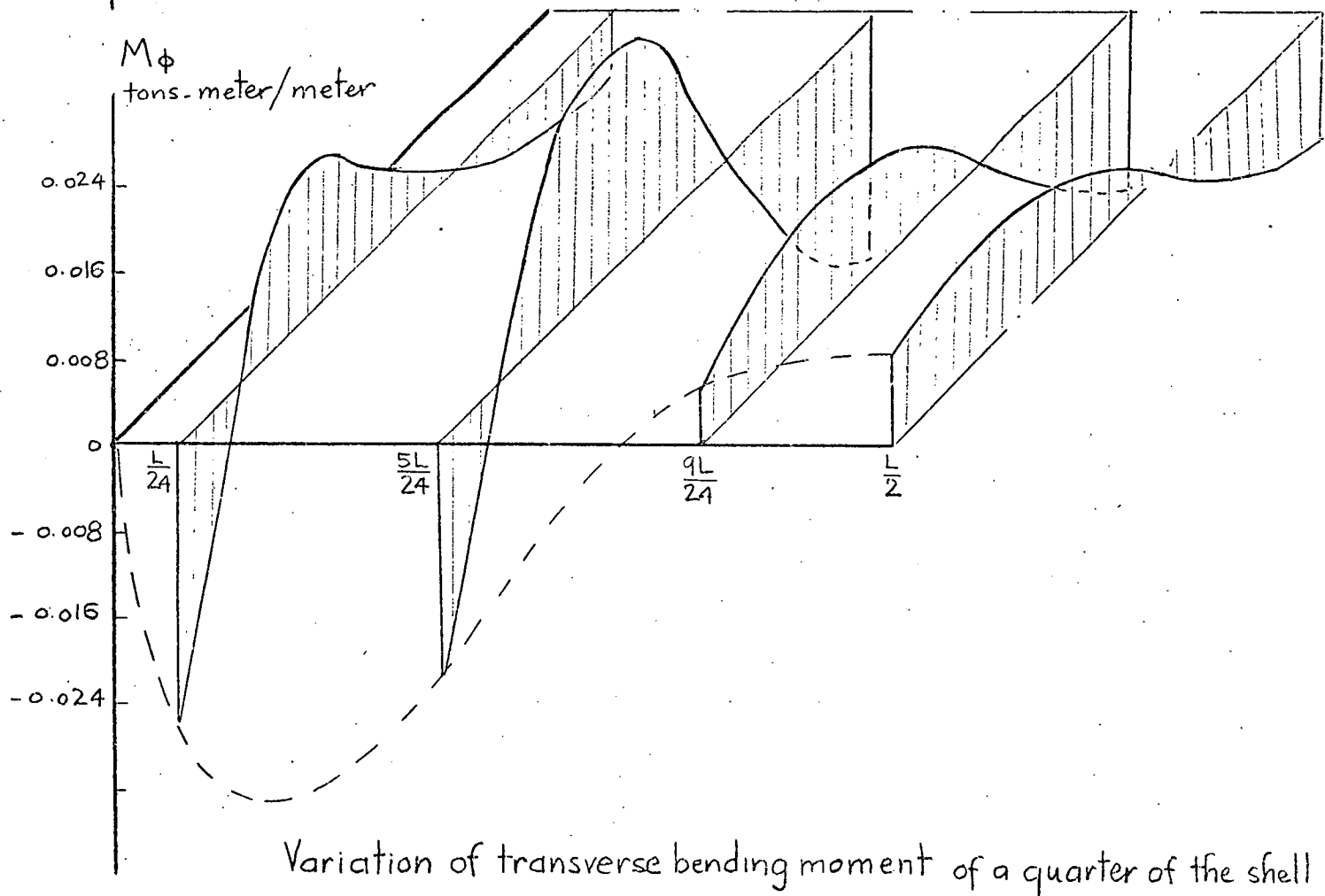
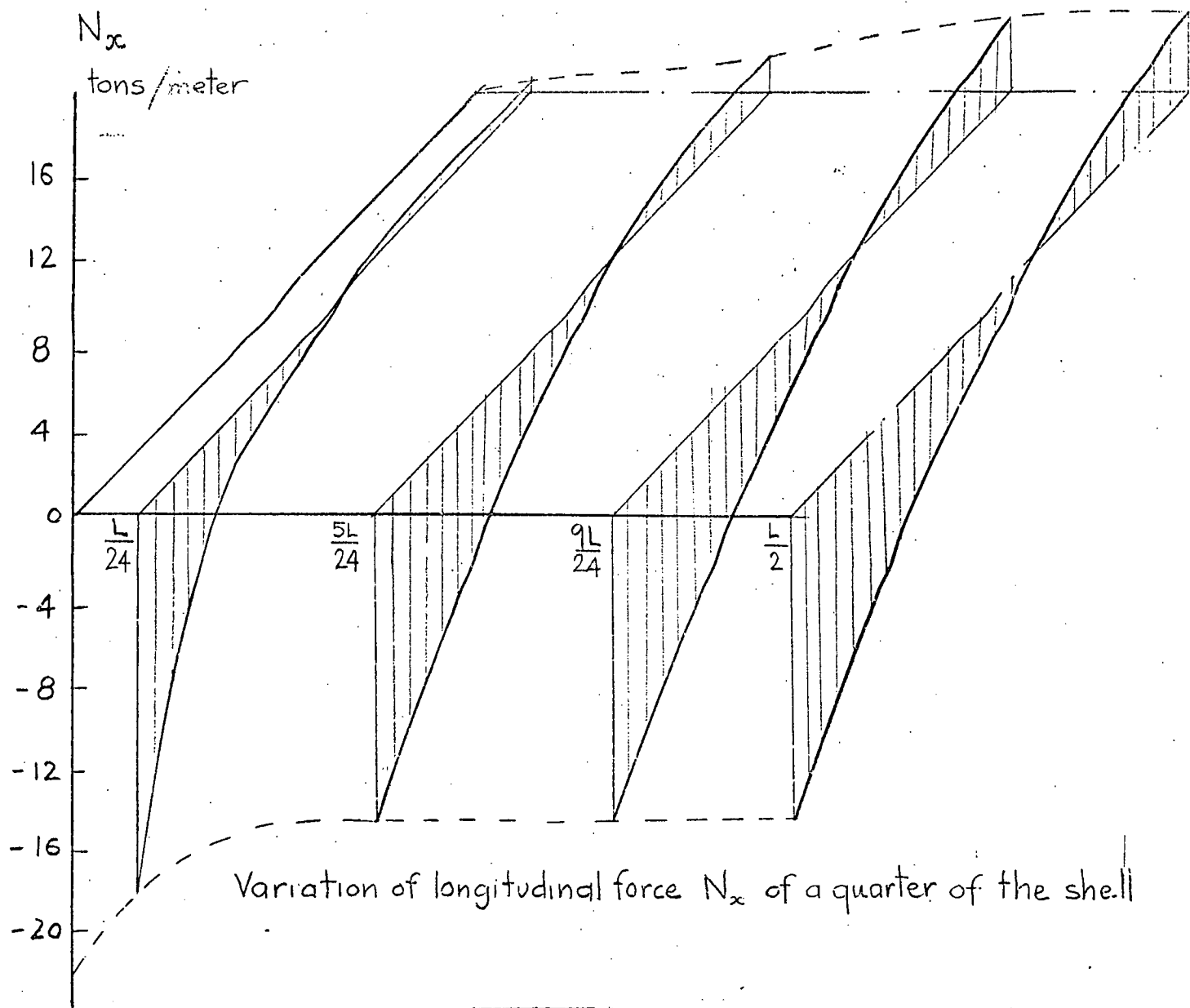
Longitudinal Stress N_x (tons/meter) along edge of shell .



In-plane shear $N_{x\phi}$ (tons/meter) along edge of shell .



Graph 3.2



Graph 3.3

Graph 3.2 shows the normal longitudinal forces and shear stress resultants along the junction of the shell and edge beam obtained by the Finite Element method, the one terms Fourier series and the five terms Fourier series calculations (Dabrowski). While Dabrowski's results show some improvements from that of the standard method (one term Fourier series), they still give inaccurate values for the part of shell near the traverse. As should be expected (and given by the Finite Element method) the normal forces and shears tend to assume maximum value near the vicinity of the traverse. It is further noted that with the increasing number of terms, the Fourier series results oscillate near the centre of the shell and diverge near the traverses.

Graph 3.3 shows the variation of longitudinal normal force N_x and transverse bending moment M_ϕ for a quarter of a shell.

From the above analysis, it is seen that the Finite Element method gives more satisfactory results than the standard Fourier series analysis for the problem of prestressed edge beam. Recently another analytical method has been developed to solve the problem more realistically. (see Reference 5).

3.6 Shell Vibration:

The present shell finite element programme can be easily extended to study the vibration problem. For free vibration, the governing equation can be written as

$$[K]\{\delta\} = -[M]\{\ddot{\delta}\}$$

where $[K]$ is the stiffness matrix
 $\{\delta\}$ is the displacement vector
 $[M]$ is the mass matrix
 $\{\ddot{\delta}\}$ is the acceleration vector

For natural vibration $\{\delta\} = \{\delta_0\} \sin \omega t$

from which $\{\ddot{\delta}\} = -\omega^2 \{\delta_0\} \sin \omega t$

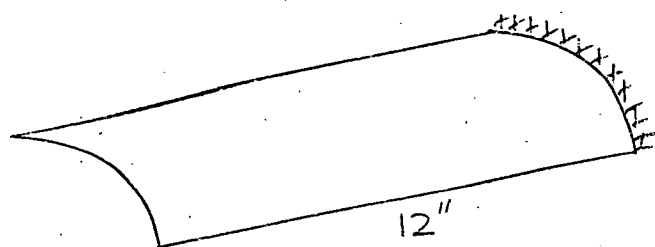
Hence $\{[K] - \omega^2[M]\}\{\delta\} = 0$

This is a typical eigenvalue problem. If (K) and (M) are $n \times n$ matrix then there will be in general n values of w for which the above equation is satisfied.

The mass matrix for a rectangular plate element has been derived by various authors (Reference 1,6) and is used here to study the vibrations of a cantilever shell.

Experimental work -

A cantilever shell was built and tested in the laboratory with Bruel and Kjar vibration equipment:

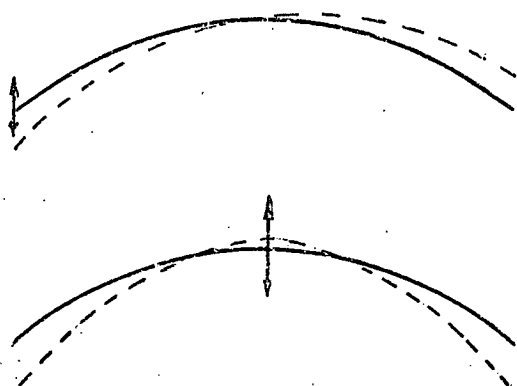


$$h = 0.0625''$$

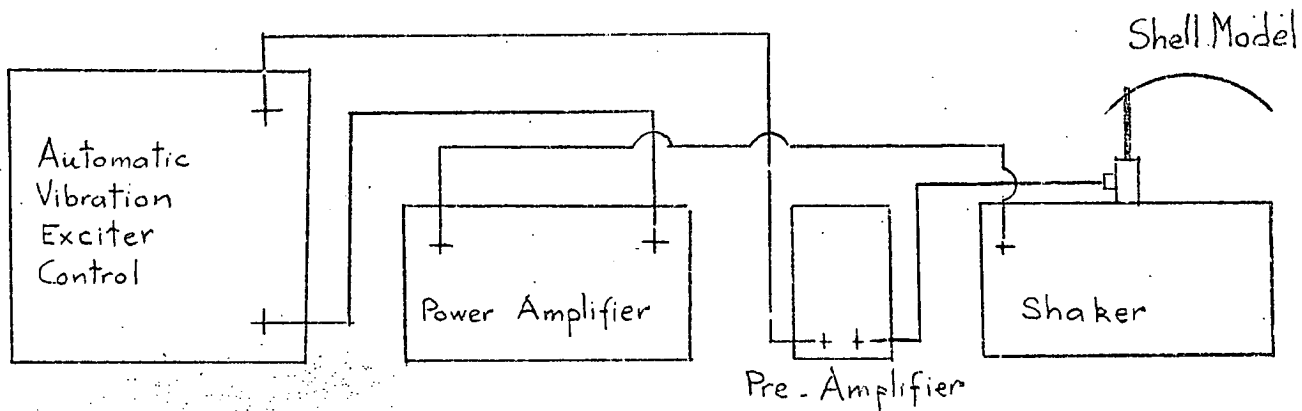
$$L = 12''$$

$$\phi_0 = 30^\circ$$

$$R = 5''$$



The shell was vibrated at the corner to stimulate the antisymmetrical mode or at the crown for the symmetrical mode.

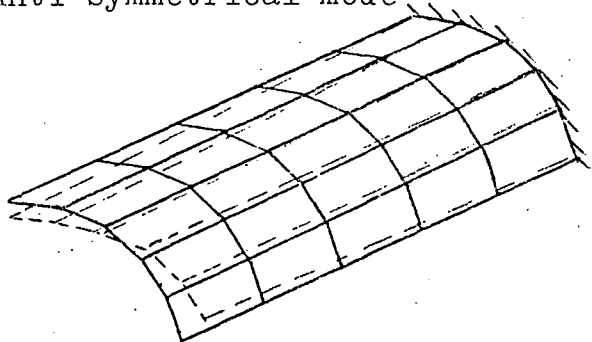


The vibration was fixed to a constant amplitude and the frequency was slowly increased, the power required to drive the shaker increased proportionately until the natural frequency of the system was reached, then there was a sudden drop in the power input. This can be observed on the power meter.

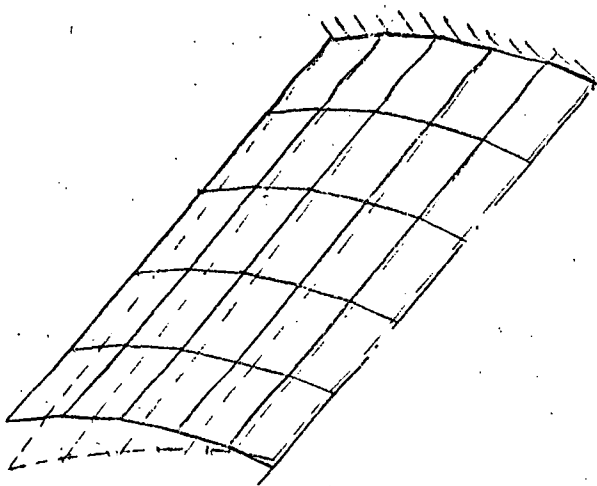
The lower natural frequency of the antisymmetrical mode determined by this method for the cantilever shell is 55 cps, and that for the symmetrical mode is 105 cps.

Finite Element Analysis:

Anti-symmetrical mode



Symmetrical mode



The shell was approximated to 25 elements. The lowest natural frequency obtained was 160 cps, with the vibration pattern

$$\{1.0, 0.6, 0.2, -0.2, -0.6, -1.0\}$$

Only half of the shell needs to be analysed due to symmetry. This half was also approximated to 25 element.

The lowest natural frequency obtained was 110 cps with the vibration pattern.

$$\{1.0, 0.75, 0.48, 0.23, 0.06, -0.007\}$$

3.7 Other Formulations:

(a) Triangular flat element - Triangular flat element has been used for shell analysis by various authors (Reference 1, 10, 11). The basic assumptions involved are the same as those for rectangular element. The main advantage of triangular element is that it can be used for shell of double curvature. For cylindrical shell it offers no special advantage except that non rectangular shapes and cut-outs can be better represented. On the other hand, triangular element program requires more elements than that of rectangular ones for the analysis of cylindrical shells.

(b) Curved rectangular element - Curved rectangular element was tried by the author. This was done through the existing model by converting the rectangular co-ordinate system to the polar co-ordinate system with similar shape functions. This programme did not give any better accuracy than that of flat

element. The reason is that the rigid body displacements have not been adequately represented. Cantins and Clough have offered a solution to this problem (reference 9). These refinements create some difficulties in programming and increase computing time.

3.8 Concluding Remarks:

The equivalence of the principle of minimum potential energy and the equations of statics as used in Finite Element method has been demonstrated. This study clarifies some aspects of the use of shape function in Finite Element analysis.

The solution of a cylindrical shell as an assembly of flat elements has been presented. This study is extended to cover two useful applications: the analysis of shells with prestressed edge beams and shell vibration. While the approximation is still rather crude, satisfactory agreement has been reached with experimental measurements. Thus, the Finite Element method can be used in design with reasonable confidence.

Appendix B presents an extension of the Finite Element method. By incorporating experimental information (Moire Fringe) into a Finite Element programme, the number of elements required for the analysis is considerably reduced without any undue loss of accuracy. This development is particularly useful in the analysis of structures with many cut-outs and irregularities where the computer storage problem becomes acute.

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CHAPTER IV

DESIGN OF CONCRETE SHELLS

4.1 Introduction;

The design of a structure is usually a trial and error process. The shape and dimensions of a structure are first determined by its functional or aesthetic requirements; the structural analysis is then carried out to check the stress distribution under various loading conditions; finally the designer must consider a suitable construction method to enable the structure to be built. The three phases of work are interconnected, difficulties in analysis can be often avoided by altering the design concept, actual construction practice determines how refined the analysis must be.

In shell design, the actual construction method is the determining factor. The savings in shell structures which stems from the low consumption of materials will be more than nullified if the cost of forming the shell is too expensive. Thus simplicity in construction will justify some increases in materials or in the complexity of the analysis.

Shells are difficult to design rationally. Rigorous mathematical analysis is often too complex to be used directly in design. Designers frequently resort to simplifications which normally give conservative designs; furthermore little experimental evidence exists to verify the mathematical analysis.

Before the advance of the computer, the results of analysis are usually presented in tables and charts upon which the design is made. This practice has a limited range of application due to restrictions imposed when the tables are prepared. Alternatively, the designer can concentrate on physical action of the shell; depending on the situation, shells can sometimes be replaced by 'equivalent' structures (such as beam, slab....).

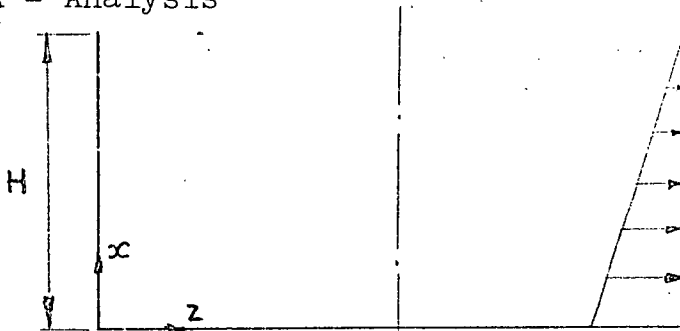
The two approaches are not mutually exclusive. Some insight into the physical action of shell is essential to obtain a good approximation upon which mathematical analysis is performed. Design by analogy also requires analysis at some stage.

In this chapter, the design considerations for liquid retaining tanks and shell roofs are discussed. Possible applications of the works in previous chapters into design practice are indicated wherever appropriate. The discussions are partly based on the author's own experience while working with a consulting firm.

4.2 The Design of Liquid Retaining Circular Tanks:

Liquid retaining circular tanks can be identified as circular thin shells with axi-symmetrical loadings. The governing differential equation has been established in Chapter I. The analysis consists of various ways of handling the equation in conjunction with various boundary conditions. The design discussion will be limited to aspects of reinforced concrete designs as these occur most often in practice.

A - Analysis



The general differential equations for this cylindrical shell with axi-symmetrical loadings is generally accepted and used as basis for most circular tank designs.

$$\frac{d^2}{dx^2} \left(D \frac{d^2 w}{dx^2} \right) - \frac{Eh}{a^2} w = p_r$$

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad h =$$

$$a =$$

$$p_r = -\gamma(H-x)$$

If the tank wall is of constant thickness, the equation becomes

$$\frac{d^4 w}{dx^4} + 4\beta^4 w = \frac{-\gamma(H-x)}{D}$$

where

$$\beta^4 = \frac{3(1-\nu^2)}{a^2 h^2}$$

The solution is

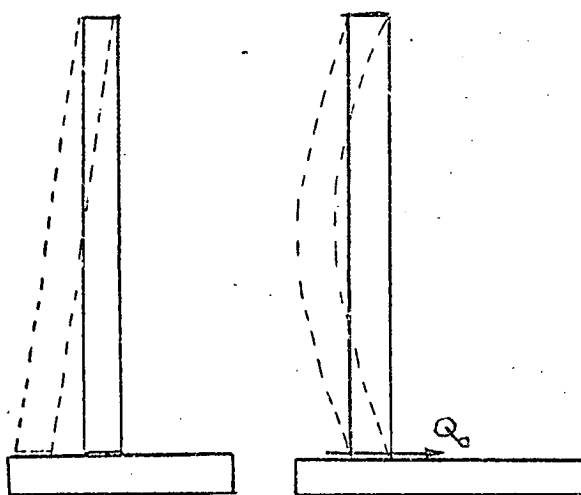
$$w = e^{-\beta x} (C_1 \cos \beta x + C_2 \sin \beta x) + \frac{\gamma(x-H)}{Eh} a^2$$

The significance of the terms has been explained elsewhere.

If the tank wall is of non-uniform thickness, the solution of the problem requires the integration of the first basic fourth order differential equation. Timoshenko² gives a solution for linearly varying wall thickness. The solution is complicated and is not suitable for design. The final results do not differ much from the constant thickness case.

Boundary Conditions: The following base conditions are normally considered:

- (a) Freely sliding base, for which the membrane theory gives all adequate solutions, i.e. $C_1 = C_2 = 0$
- (b) Hinged base, i.e. the base is free to rotate but not to slide; a radial shear correction is needed together with the membrane solution.



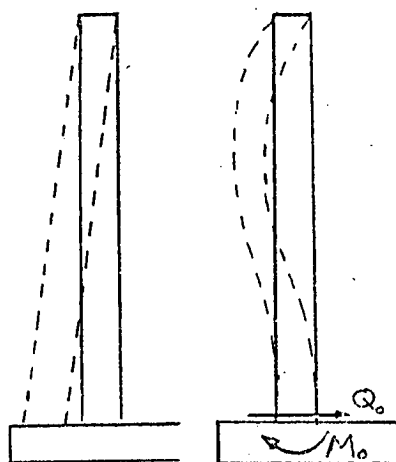
Membrane solution gives a radial deflection at the base $w = \frac{\gamma a^2 H}{Eh}$.

A shear Q_0 must be applied at the base so that radial displacement at the base is zero.

$$Q_0 = 2\beta^3 D \frac{\gamma a^2 H}{Eh}$$

(See Chapter I - Section 1.5)

- (c) Fixed base, i.e. neither displacement nor slope is allowed at the base; both moment and shear correction are required.



$$(w)_{x=0} = -\frac{1}{2\beta^3 D} (\beta M_0 + Q_0) - \frac{\gamma a^2 H}{Eh} = 0$$

$$\left(\frac{\partial w}{\partial x}\right)_{x=0} = \frac{1}{2\beta^3 D} (2\beta M_0 + Q_0) + \frac{\gamma a^2}{Eh} = 0$$

(d) Partially fixed base - If the tank floor is built monolithically with the tank wall, then the base of the tank is only partially fixed. Rotation of the base slab must be considered. This problem is treated rigorously by Timoshenko and approximately by Billington.

(e) Partially sliding base - Some shear and some moment occur but not to the extent of (b) or (c). This condition is considered to obtain a more realistic design. The method relies heavily on the engineer's judgements.

Relative merits of various base conditions will be discussed in a subsequent section.

Methods of Calculation - Some methods of calculation have been developed based on the beam or elastic foundation analogy.

A graphical method has been developed by Mterisch⁵ using an assumed deflection curved to calculate the moment diagram from which a new deflection curve is obtained; the process is repeated until satisfactory convergence is achieved.

⁶
A numerical method has been developed by writing the equation in finite difference form to solve it as a set of linear equations. This scheme is only feasible if a computer is used. Various conditions of loading, fixity and movements of the base and the top of the tank can be taken into account easily.

For wall of constant thickness, tables and charts have been prepared for various ratio of diameter and height by the Portland Cement Association.⁸ Design from tables is so simple that it becomes the standard design procedure.

Design Problems:

(1) 'Thick Walled' vs 'Thin Walled' design:

The Code of Practice for the Design and Constructions of Reinforced Concrete Structures for the storage of water and other liquids (CP 2007 - 1960) on which most of British designs were based states that ".....the tensile stress in the concrete shall not attain that at which it may fail in tension, with some small margin of safety". Thus the objective of the design is to provide enough hoop steel so that if the wall cracks the hoop steel will be capable of resisting the entire hoop force.

On the other hand, the wall must not crack, therefore concrete hoop stress must be below the cracking stress given in CP 2007. (175 psi). This consideration often gives "thick walled" sections (e.g. 2'6" to 4'0" for a 2 million gallon water tank 26' high).

The Portland Cement Association follows the same objectives, however the concrete stress is allowed to rise to 300 psi (for 3000 psi concrete). It also points out that lowering the allowable steel stress actually tends to make the concrete crack because the higher steel area required increases the concrete shrinkage stress. It is also clear that the bond quality is of major importance in regard to width of cracks and leakage. Therefore, reinforcing bars should be deformed and of smallest possible size. (Allowable steel stress 10,000-14,000 psi for ring steel).

The South Australia Engineering and Water Supply Department advocated the "thin walled" design philosophy⁷ in which:

- tensile concrete stress is neglected
- higher steel stress is adopted (25,000 psi)
- crack control factor is considered.

From this point of view, a crack is not considered as undesirable if it does not produce leakage. This can be achieved by:

- the autogenous healing of cracks which are caused by the formation of crystals of calcium carbonate and hydroxide in the crack from the water

- crack control - crack width can be calculated and limited to a certain maximum so that the autogenous healing process can take place. The works of the European Concrete Committee and Kaar & Mattock (USA) produce some formulas for the calculations of crack width.

(II) Types of Restraints at Base:

Early design assumed that the base is fixed. This will, of course, give the most economical design for the tank wall, but the base is seldom actually fixed. It is difficult to predict the behaviour of the subgrade and its effect upon the restraint at the base. Therefore it is more reasonable to assume the base is hinged. Some other advantages of the hinged base are:

- The compression face of the concrete is on the water side of the tank; this eliminates horizontal cracks at the water face and the Poisson's ratio effect reduces the magnitude of any vertical cracks that may occur.

- Size of the footing is smaller. A footing ring is normally required for the base; some radial deflection will occur and should be allowed for. The base ring will deflect the same amount as the base of the wall and consequently they will take some steel stress. For a fixed or hinged base this deflection must be kept very small, therefore a big ring is required and the amount of steel in the ring is used uneconomically. The solution is to make the base hinged and partially sliding. The base shearing stress can be determined by subtracting the wall resistance from the liquid load and the amount of steel necessary to carry the remainder in the base ring can then be determined.

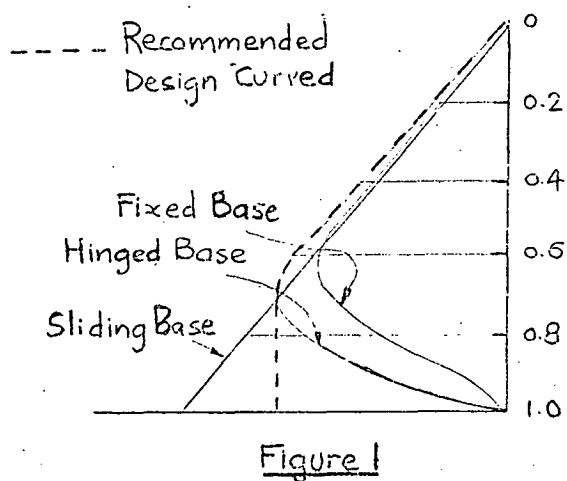


Fig. 1 gives the ring tension for the bases that are fixed, hinged or sliding. The upper half of the wall is affected little by the base condition. Maximum ring tension of the hinged base is larger than that of the fixed base (about 20%) but the latter procedure gives

a more realistic distribution of steel. To reduce the size of the footing, a partially sliding base is advocated. Thus for overall economy, the ring tension is designed as given by the dotted curve.

For a 'thin walled' tank, a free base wall should be used as the hinged base tank will lose all the advantage of higher allowable hoop stress. Free base wall also eliminates vertical bending, membrane theory alone is sufficient for design purposes and hoop reinforcement is operating at maximum efficiency.

(III) Other Problems:

(a) Prestressed Concrete Cylindrical Tanks - The object of prestressing is to offset the ring tension arising from liquid loading and to prevent leakage. The main concrete shell is then at its minimum stress condition under full load. The problem of the base is similar to that of a reinforced concrete design. Prestressed concrete tanks have thinner walls than that of normal reinforced concrete but the construction cost is higher. Small capacity tanks (under 25,000 gallons) are not worth prestressing. The problems of prestressed tanks have been fully discussed by Creasy⁹.

(b) Settlement of foundation - If the tank is supported on the ground, the effect of settlement of the foundation must be considered. Lightfoot¹⁰ considered the problem when the reactive pressure is uniformed as with a plastic soil. The problem of tank on elastic foundation can be tackled with the use of Schorer's equation written with longitudinal displacement 'u' as the main variable.

These problems are not normally considered in tank design when the footing foundation is reasonably firm and even.

(IV) Construction Problems:

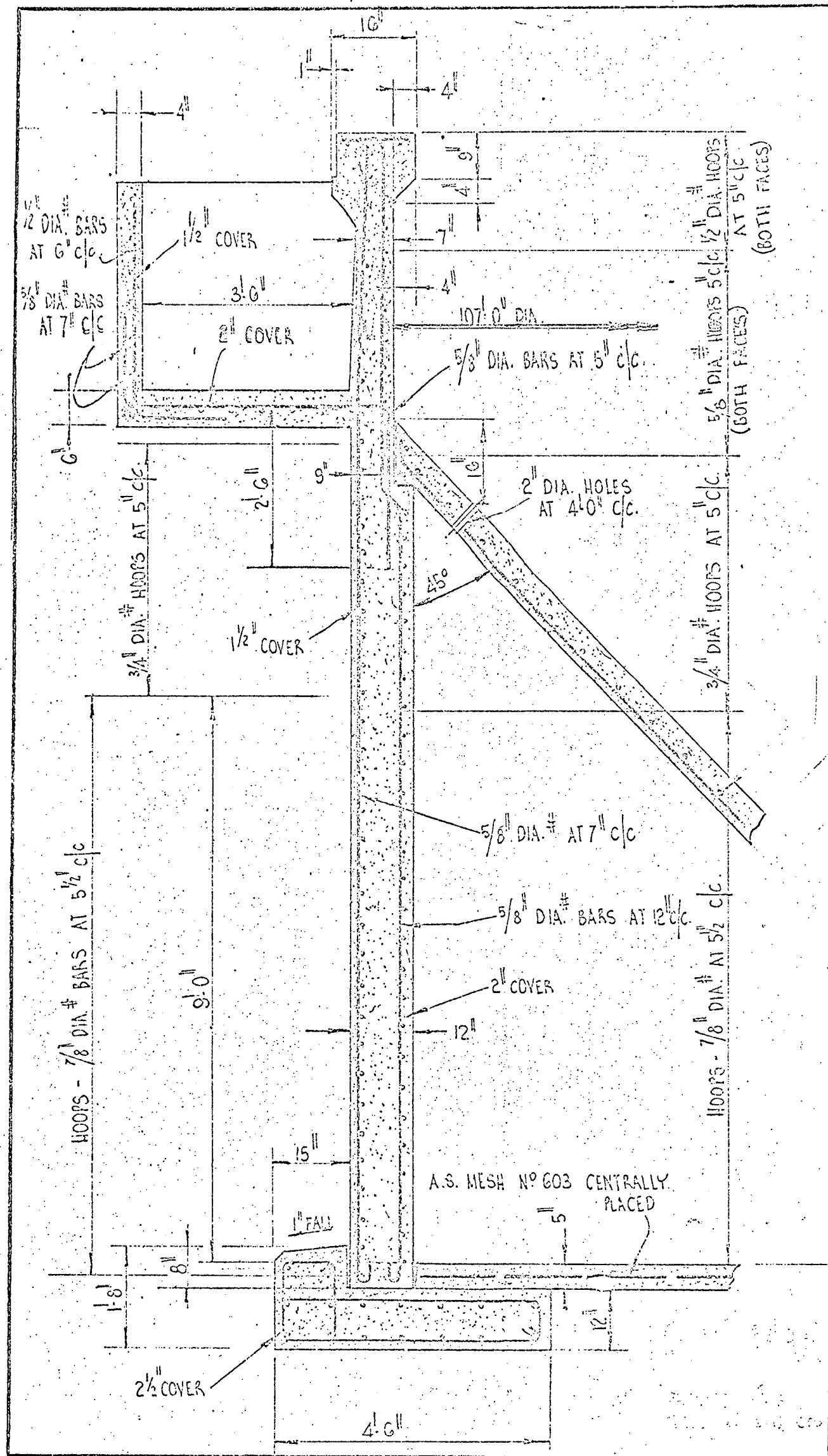
The general principles of concrete construction practice apply, but emphasis is placed on watertightness. Joints are potential source of leakage and their number should be minimised. To ensure watertightness, a water-stop made of sheet copper^{or} rubber is normally placed across the joint. For a sliding base, a groove is normally provided at the inside of the wall and the base junction and filled up with tar or asphalt. Vertical joints are preferable to horizontal ones as they require less attention to ensure watertightness.

The structure is necessarily monolithic, concrete should be placed in horizontal lifts not over two feet, no temporary joints should be allowed.

Curing is of prime importance especially for tank walls because it reduces shrinkage stresses, increases concrete strength and reduces cracking tendencies.

(V) Use of Concrete Tank for Storage of Other Liquids:

Industrial liquid - The liquid stores must not be injurious to the concrete - Normally bases and compounds of



Wall and base of a clarifier tank designed according to the recommended design curve of Section 4.2
(Courtesy of Gutteridge, Haskins and Davey.)

carbon are not while acids and many salts are. Protective measures include lining of tank with glass plate or coating the tankwall with an acid resisting surface (asphalt, wax, paraffin).

Fuel Oil - Loss will occur due to the percolation of the liquid through the concrete. A double walled reservoir is normally used. Cavities between the walls and floor are filled with water which keeps the tank petrol tight. Alternatively tank wall is lined with aluminium sheet.

4.3 The Design of Shell Roofs:

A shell roof offers a means of covering a large unobstructed area with a minimum amount of materials. This solution, unfortunately, is not always the most economical. The cost of formwork and labour usually offsets the savings in material. Consideration about construction techniques should be in the mind of the designer right from the beginning of the project.

A - General Considerations:

To cover a certain area with a shell roof, the following alternatives are usually considered:

- Continuous short shell (Figure 2-a)
- Multiple long shell (Figure 2-b)

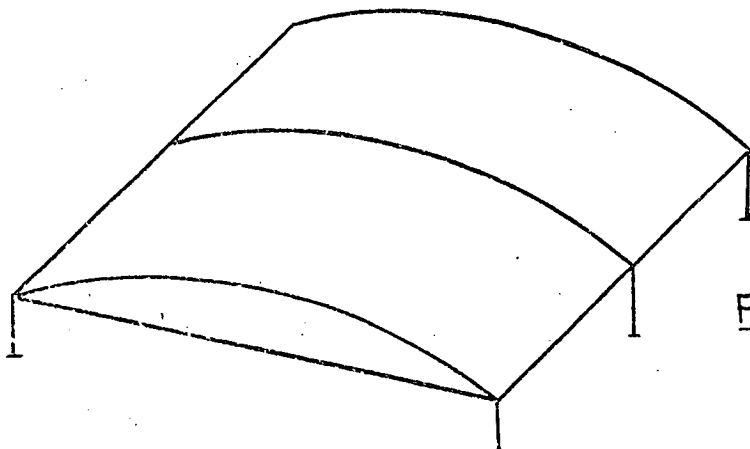


Figure 2-a.

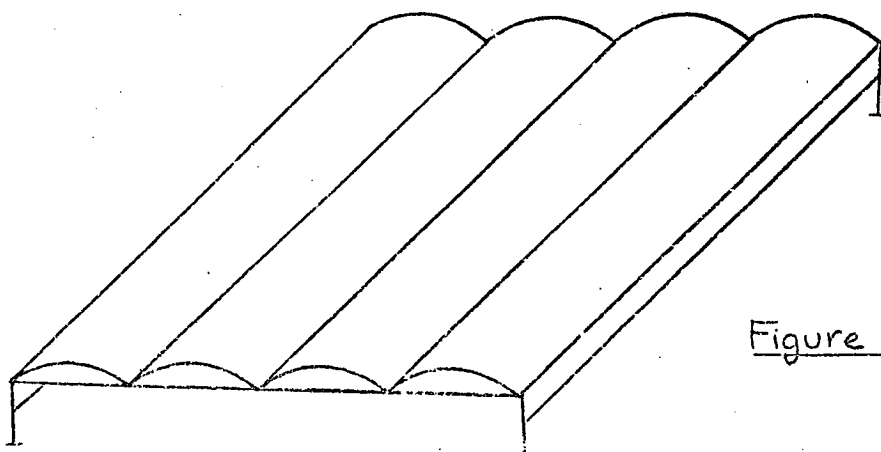


Figure 2-b.

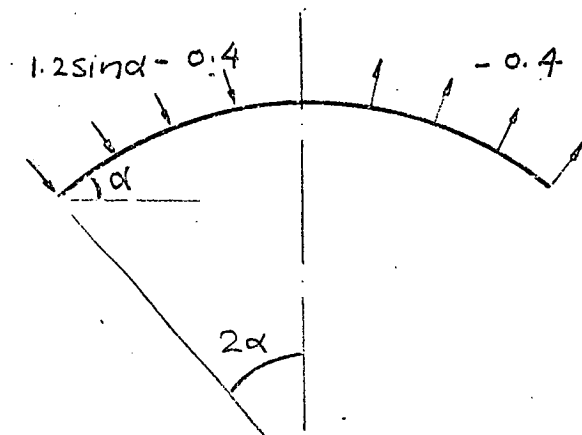
Short shells can have a chord width of up to 400 ft. with spans between one third and one sixth of the width.

Long shells can have a span up to 150 ft. - for spans greater than 150 ft., a break is needed to allow for temperature expansion. For reinforced concrete a practical span limit is about 100 ft., longer shells need prestressed edge beams to be economical.

The shell angle is normally restricted to $30-45^{\circ}$ for two reasons:

- Only internal formwork is required, this facilitates the placing of concrete and reinforcements.

- Wind load on shells in this range can be safely ignored. According to S.A.A. Int. 350, the wind load factor formula is $1.2\sin\alpha - 0.4$. Thus the whole shell will be under



suction if the shell angle is less than about 40° .

Shell Loadings: The following loadings are usually considered:

- Dead Load which depends on shell thickness.
- Live Load to allow for light access to the roof i.e. about 15 psf.

With most shells of roof thickness, about 4 ins., giving a total uniformly distributed load of 55 psf, together with 200 lbs. concentrated load at any point on the shell and any other unusual loadings.

Shell thickness - The main design load on shell is usually its dead weight (about 75%), thus the shell should be as thin as practicable. A decrease in thickness will decrease the dead weight which is proportional to the amount of materials (concrete and steel) required. Thin shell also has the advantage that any edge effect is likely to confine to a region near the edge only.

Australian and American concrete codes do not have a

specific clause for shell thickness, but they recommend a minimum thickness for slab of about 4 ins. The European code allows a thinner section of about 7.8 cm (3 ins.). However thinner shells have been built elsewhere.

B - Shell Design:

(i) Accuracy of calculations - Concrete is usually placed with a tolerance of $\frac{1}{4}$ " on dimensions. Assuming that the reinforced shell behave homogeneously, the tolerance on stress calculation can be estimated.

- Direct stresses vary directly with thickness

Allowable variation in stress is

- Bending stresses vary cubically with thickness

Allowable variation is approximately 20%.

Thus an overall accuracy in calculations of about 10% should be satisfactory.

(ii) Stresses in shells - Analysis in Chapter I showed that there are in general three types of stresses; membrane stresses, edge effects (bending stresses caused by edge deformation) and stresses caused by the phenomenon of inextensional deformation. For a well designed shell, the membrane stresses should carry most of the load, the edge effects should be confined to a small region near the edges and the inextension deformation should be eliminated as much as possible.

(iii) Preliminary design - An initial choice of dimensions is made so that the concrete stresses will fall within the allowable limits. For short shells membrane theory is used, for long shell, beam theory is suitable. As shown in Chapter I, these theories are simple, all calculations can be done quickly.

(iv) Bending Analysis - A thorough analysis of the shell should be carried out after preliminary dimensions have been chosen to obtain a better estimation of the stresses and hence the design of reinforcement can be completed.

- Analytical method: The total solution can be obtained by solving the differential equation. Schorer's theory is

suitable for long shells. Donnell's theory is used for short shells. Beam arch theory is used when the shell is unsymmetrical or the cross-section is not circular. All these methods of analysis can be programmed for electronic digital computers. For shells of sizes within common limits, tables and charts have been produced in shell literature and they can be used directly.

- Finite Element Method: As shown in Chapter III the Finite Element method can be used for shell analyses with the possibility of extension to include other structural members such as edge beams and traverses. The main problem here is the ready access to a reasonable large storage capacity computer to handle the large number of equations to be solved. Cost and time of such analysis is another factor to be considered. The development of the Finite Element method as used for shell is still at an early stage, further improvements could be expected in the future.

- Model Test: With the technique developed in Chapter II, model analysis could be used as an aid to design. Model tests are usually regarded as expensive, the merit of the Moire' Ligtenberg method as used for shell is that a lot of information can be obtained from relatively simple, inexpensive models. If strain gauge readings are also taken, a complete picture of the state of stress can be obtained. Model tests can also be used to substantiate the analysis carried out by Finite Element or any other method.

(v) Boundary Conditions - Boundary conditions statements are needed for both closed form analytical methods and numerical methods. Some of the boundary conditions often met in practice are discussed here. The symmetry of the structure and loading is often used to reduce the size of the problem.

Boundary conditions along a free edge: For short shells, a free edge is possible. The boundary conditions for such an edge are obviously:

- (a) No transverse moment, i.e. $M_\phi = 0$
- (b) No transverse shear, i.e. $Q_\phi = 0$
- (c) No transverse normal stress, i.e. $N_\phi = 0$
- (d) No shear along the edge, i.e. $N_{x\phi} = 0$

- Boundary conditions along an edge beam: The edge beam is usually deep and narrow. It is assumed that it possesses stiffness to resist torsion or bending out of its plane.

The rigorous boundary conditions are:

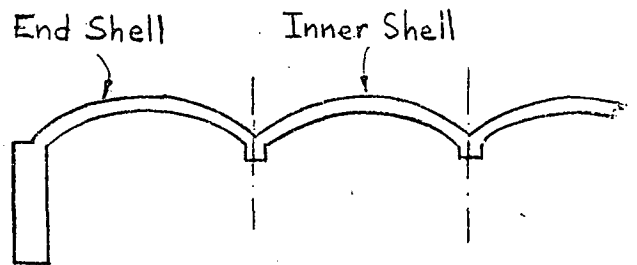
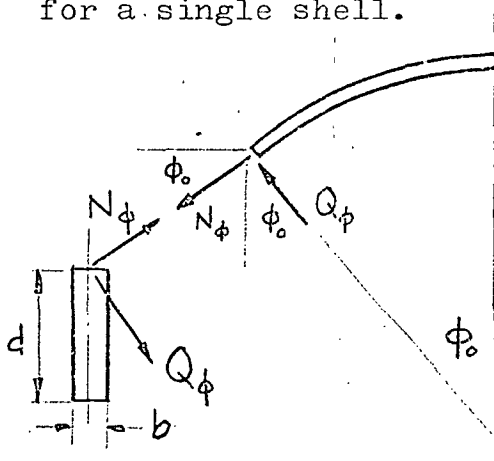
- | | |
|-----------------------------------|--|
| (a) The longitudinal displacement | } of the shell edge and the edge beam shall be the same at their junction. |
| (b) The vertical deflection | |
| (c) The horizontal deflection | |
| (d) The rotation | |

Subject to the above assumptions, conditions (c) and (d) are often replaced by -

- No transverse moment, i.e. $M_\phi = 0$

- No lateral thrust, i.e. $H = N_\phi \cos \phi_0 + Q_\phi \sin \phi_0 = 0$

for a single shell.



For a symmetrical inner shell of a multiple shell group, no horizontal deflection or rotation exists at the edge due to its symmetry.

- Boundary conditions along a traverse -

The end traverse is usually assumed to be rigid in its own plane but completely flexible in the direction normal to its plane. The boundary conditions for these assumptions are:

- | | | |
|------------------------------------|--------------|---|
| (a) No radial deflections, i.e. | $w = 0$ | ; |
| (b) No tangential deflection, i.e. | $v = 0$ | ; |
| (c) No longitudinal moment, i.e. | $M_x = 0$ | ; |
| (d) No transverse moment, i.e. | $M_\phi = 0$ | ; |

These boundary conditions are necessary for the theories developed in Chapter I to be used; since only then is it possible to express all the parameters in terms of Fourier's series. The method will break down for other boundary conditions.

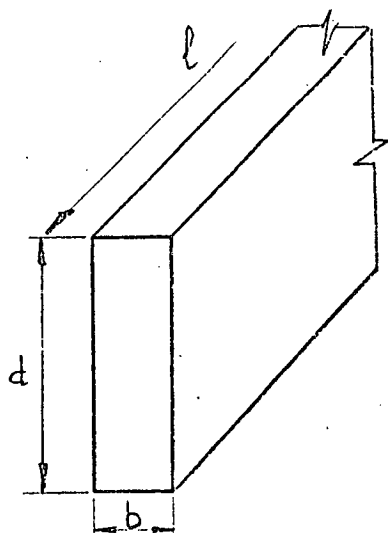
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C. - The Design of Edge Beams:

Edge beams are introduced at the edges of long shells to reduce the shell forces and moments. The edge beam width is usually from 2 to 3 times the shell thickness with a minimum of 5 or 6 ins. Its depth is chosen so that the shell will have adequate structural depth (between one sixth and one twelfth of the span). These dimensions are usually selected at the same time as those of the shell using beam theory.

Analysis of Edge Beam:

The interaction between the shell and the edge beam causes the latter to deflect vertically, laterally and twist. As mentioned in the discussion of shell boundary conditions, the beam is assumed to have no resistance to horizontal thrust and twist. For a deep and narrow beam, these assumptions involve little error.



Forces acting on the edge beam are those from the shell edge and the dead weight of the beam:

Vertical force from shell

$$N_{\phi} \sin \phi_0 - Q_{\phi} \cos \phi_0$$

Dead weight of beam W expressed as a Fourier series

$$W = W_0 \cos \frac{\pi x}{l} + \dots$$

Shear $N_{x\phi}$ acting on the top of the beam. This shear force can be replaced by a central force and a moment

$$\begin{aligned} \text{Central force} &= \int_0^{l/2} N_{x\phi} \sin \frac{\pi x}{l} dx \\ \text{Bending moment} &= \frac{l}{\pi} \frac{d}{2} N_{x\phi} \end{aligned}$$

Thus, the stress on the top of the edge beam can be estimated

$$\begin{aligned} (\sigma)_{z=\frac{d}{2}} &= \frac{l}{\pi} \frac{N_{x\phi}}{A} + \frac{l}{\pi} \frac{d^2}{4I} N_{x\phi} + (N_{\phi} \sin \phi_0 - Q_{\phi} \cos \phi_0) \left(\frac{l}{\pi} \right)^2 \frac{d}{2I} \\ &\quad + W \left(\frac{l}{\pi} \right)^2 \frac{d}{2I} \end{aligned}$$

Displacements u and v of the beam can then be computed.

$$u = \left(\frac{l}{\pi}\right)^2 \frac{N_{x\phi}}{AE} + \left(\frac{l}{\pi}\right)^2 \frac{d^2}{4EI} N_{x\phi} + [N_{x\phi} \sin \phi_0 - Q_{\phi} \cos \phi_0] \left(\frac{l}{\pi}\right)^3 \frac{d}{2EI} - W_0 \left(\frac{l}{\pi}\right)^3 \frac{d}{2EI}$$

$$v = [N_{\phi} \sin \phi_0 - Q_{\phi} \cos \phi_0] \left(\frac{l}{\pi}\right)^4 \frac{1}{EI} + N_{x\phi} \left(\frac{l}{\pi}\right)^3 \frac{d}{2EI} - W_0 \left(\frac{l}{\pi}\right)^4 \frac{1}{EI}$$

These displacements are to be matched with those at the edge of the shell to achieve geometrical compatibility.

Shells with prestressed edge beams:

If the span of the shell is long, the tensile stresses in the edge tend to become very high and deflections become excessive. Difficulty in the placing of the reinforcement may occur due to the large steel area required. Prestressing offers the following advantages:-

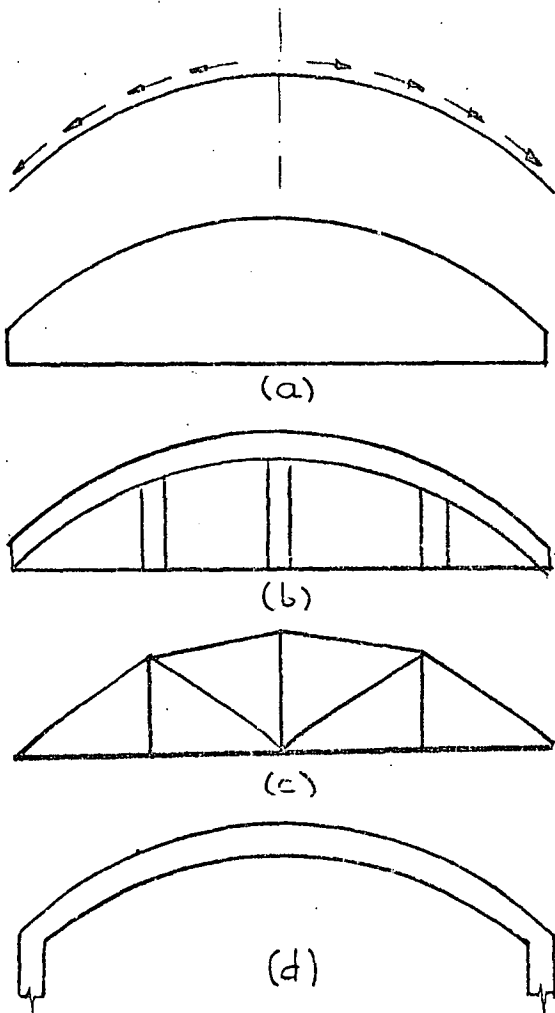
- Lower steel consumption
- Reduction of transverse moment
- Reduction of cracks due to shrinkage and temperature
- Reduction of deflection

Analysis of a prestressed beam is similar to that of normal beam. The approximate prestressing force and eccentricity must be put in a form suitable for Fourier series expansion. This practice leads to some serious difficulties as the series diverge near the traverse. Analysis of a shell with prestressed edge beam by standard methods and Finite Element method has been presented in Chapter III.

D - The Design of traverses:

The traverse is designed so that it will fulfill the assumptions made on the boundary of the shell. Ideally, the traverse should be rigid in its own plane and completely flexible in the direction normal to it. This is physically impossible; what is usually done is to ensure that the traverse will deform a negligible amount in its own plane and that it be kept as flexible as possible in the other direction without causing buckling.

The only forces on the traverse are the in-plane shear $N_{x\phi}$ and the transverse shear Q_x from the shell edge.



These forces can be resolved into vertical and horizontal components acting on the traverse.

$$V = N_x \sin \phi + Q_x \cos \phi$$

$$H = N_x \cos \phi - Q_x \sin \phi$$

The traverse is designed to withstand these loadings as well as its own weight.

Various types of traverse are used.

- Solid diaphragm (a)
- Arch (b)
- Truss (c)
- Portal frame (d)

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E - The design of reinforcement:

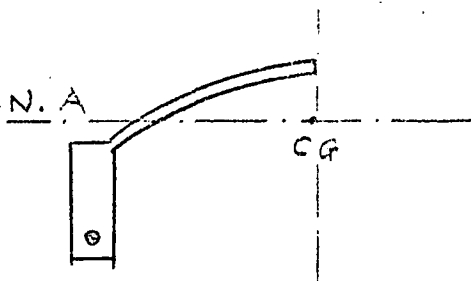
Reinforcement is required for resisting tensions arising from the following:

- Shell deformations due to loadings
- Shrinkage and temperature effects; both of these are quite high due to the large exposed area of the shell compared with the volume of concrete.
- Stress concentration due to concentrated loads, cut-outs etc.

The reinforcing bar size should be kept as small as practicable. To keep the shrinkage stress down, a fine mesh of thin bars is preferable to a coarse net of thicker bars. The use of thin bars has the disadvantage of high labour cost, they are also easily bent out of place by accident. Welded wire mesh can also be used, they are particularly suitable for cylindrical shell, they stay in place better and also reduce the labour required.

Reinforcing requirements can be conveniently grouped into three types:

- (i) Longitudinal reinforcement for longitudinal stress N_x . A long shell with deep edge beams behaves like a beam with the neutral axis lying close to the junction of shell edge and edge beam.



Thus, most of the tensile forces concentrate in the edge beam and reinforcement should be placed at the bottom of the beam. For thin and narrow

beams, more than one layer may be required.

For short shells, there may be various tensile zones in the shell. The amount of steel required is derived on the basis that it must carry all the tension. Compressive forces due to N_x need only nominal reinforcement since the concrete can take most of the compressive force. Nominal reinforcement in the longitudinal direction is also required for the whole shell surface to take temperature and shrinkage stresses.

(ii) Transverse reinforcement is required for transverse normal stress N_ϕ and transverse moment M_ϕ . They are designed as normal reinforced concrete sections, subject to combined axial load and moment.

(iii) Diagonal tension reinforcement is required to take the combination of stresses in both directions N_x and N_ϕ and the inplane shear $N_{x\phi}$. Mohr's circle can be plotted for each point on the shell and contours of principal stresses can be determined. A reinforcement scheme based on this will require the least amount of steel, however the scheme is not practical due to the difficulty and cost in achieving correct placing of the steel. Thus it is better to place the steel in a rectangular net (fabric mesh can be used) and add extra steel in the region of high diagonal tensile stress.

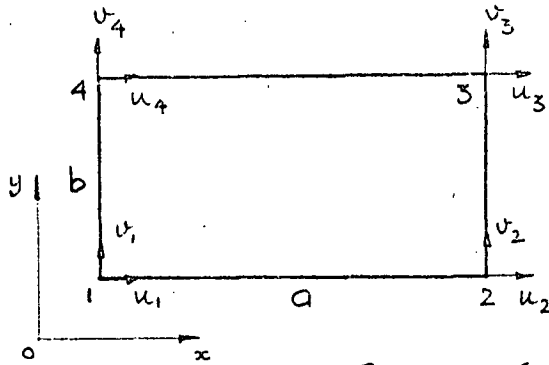
At the traverse $N_\phi = N_x = 0$, only $N_{x\phi}$ exists. By a Mohr's circle analysis, the diagonal tension is at 45° to the longitudinal axis and equal to $N_{x\phi}$.

At the centre $N_{x\phi} = 0$, thus the principal stresses are N_x & N_ϕ - no diagonal tension exists. Thus only the corner region near the traverse needs extra reinforcement to carry the diagonal tension.

The contours of slopes obtained by the technique developed in Chapter II gives a further guide to the design of reinforcement in unusual situations.

APPENDIX A .

Derivation of the stiffness matrix for a rectangular element
in plane stress problem from statical consideration .



Element size $a \times b$.

Degree of freedom per element 8

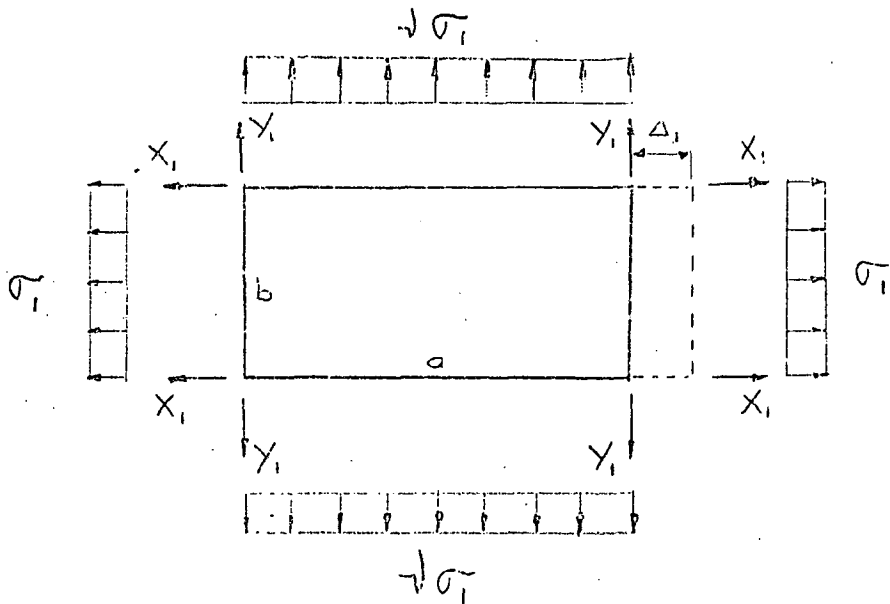
Stiffness coefficients are found
by proper combination of the
five basic stress conditions :

$$\epsilon_x = 1 ; \epsilon_y = 1 ; \gamma_{xy} = 1 ; K_x = 1 ; K_y = 1 ;$$

and three rigid body movements : two translations in x and y directions
and a rotation about the axis normal to the element plane.

For simplicity thickness of the element is taken as unity.

Straining in x direction .



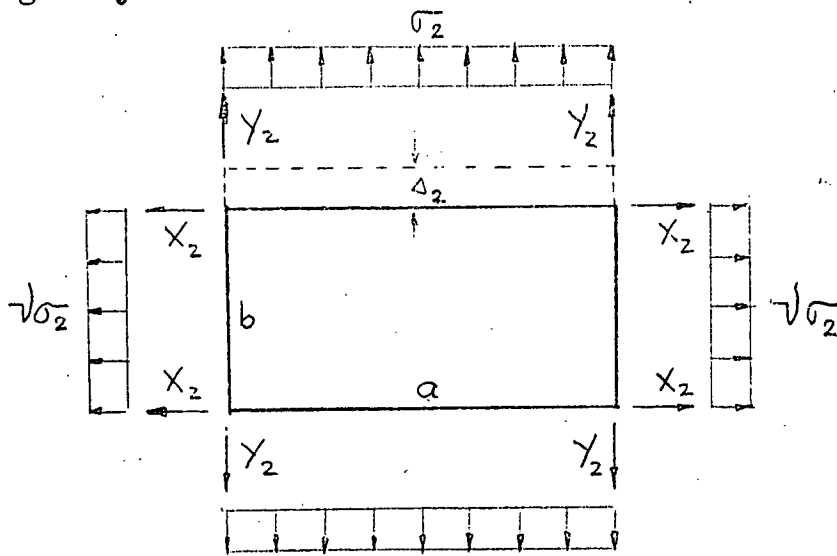
$$\epsilon_x = \frac{\Delta_1}{a} ; \quad \sigma_1 = \frac{E}{1-\nu^2} \epsilon_x = \frac{E}{1-\nu^2} \cdot \frac{\Delta_1}{a} .$$

$$\Delta_1 = \frac{1-\nu^2}{E} a \cdot \sigma_1 .$$

$$2X_1 = b\sigma_1 ; \quad X_1 = \frac{E}{1-\nu^2} \cdot \frac{b}{a} \cdot \frac{\Delta_1}{2} .$$

$$2Y_1 = a \cdot \nu \sigma_1 ; \quad Y_1 = \frac{\nu E}{1-\nu^2} \cdot \frac{\Delta_1}{2} .$$

Straining in y direction.



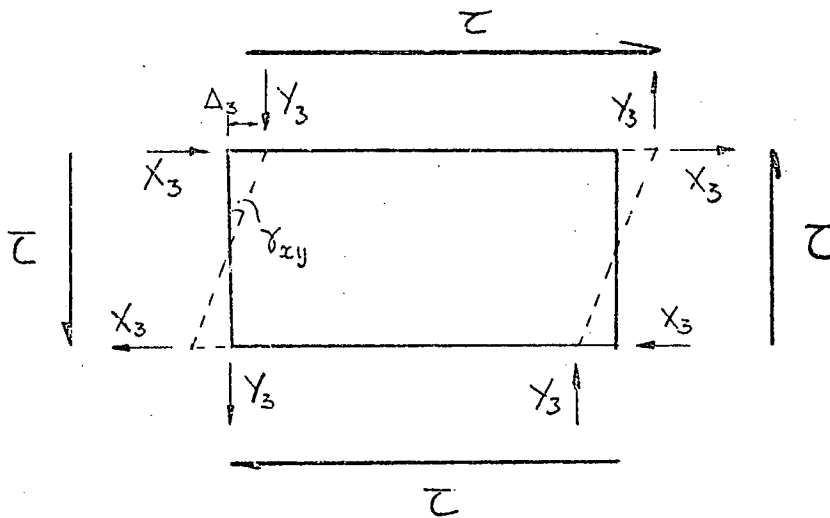
$$\epsilon_y = \frac{\Delta_2}{b} \quad ; \quad \sigma_2 = \frac{E}{1-\nu^2} \cdot \frac{\Delta_2}{b}$$

$$\Delta_2 = \frac{1-\nu^2}{E} \cdot b \cdot \sigma_2$$

$$2X_2 = b\nu\sigma_2 \quad ; \quad X_2 = \frac{\nu E}{1-\nu^2} \cdot \frac{\Delta_2}{2}$$

$$2Y_2 = a\sigma_2 \quad ; \quad Y_2 = \frac{E}{1-\nu^2} \cdot \frac{a}{b} \cdot \frac{\Delta_2}{2}$$

Shearing strain.



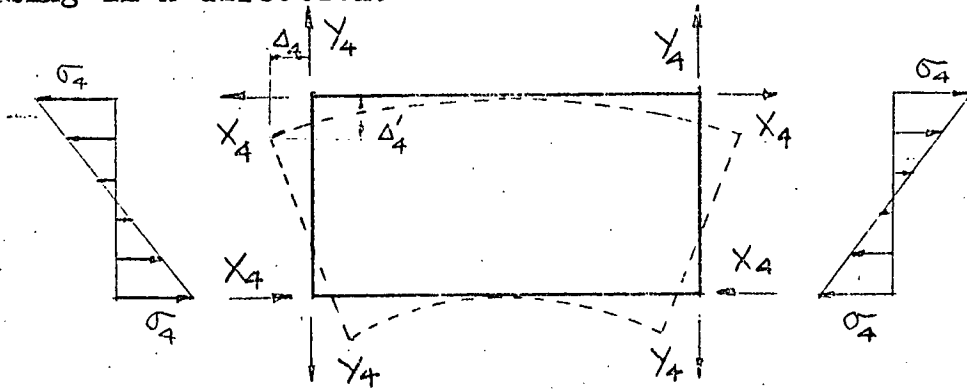
$$\gamma_{xy} = \frac{2\Delta_3}{b} \quad ; \quad \tau = G\gamma_{xy} = \frac{E}{2(1+\nu)} \cdot \frac{2\Delta_3}{b}$$

$$\Delta_3 = \frac{1+\nu}{E} \cdot b \cdot \tau$$

$$2X_3 = \tau \cdot a \quad ; \quad X_3 = \frac{E}{1+\nu} \cdot \frac{a}{b} \cdot \frac{\Delta_3}{2}$$

$$2Y_3 = \tau \cdot b \quad ; \quad Y_3 = \frac{E}{1+\nu} \cdot \frac{\Delta_3}{2}$$

Bending in x direction.



$$\epsilon_4 = \frac{2\Delta_4}{a} = \frac{\sigma_4}{E}$$

$$\Delta_4 = \frac{1}{2E} \cdot a \cdot \sigma_4$$

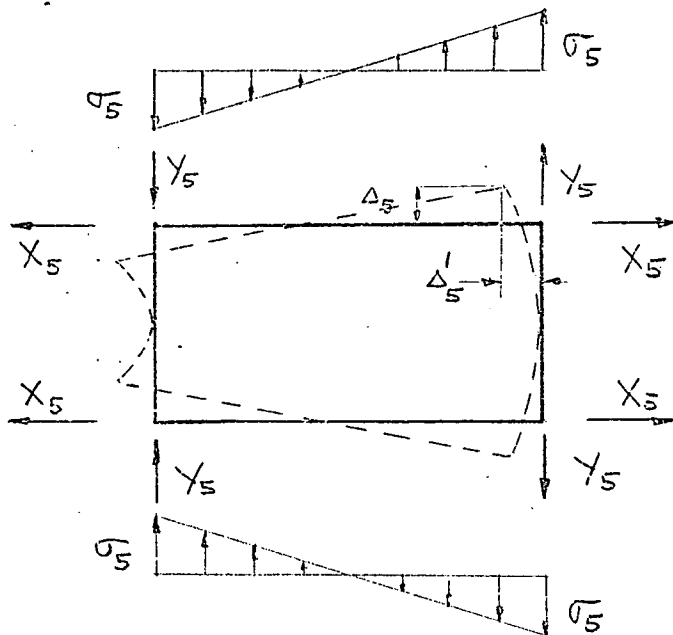
$$\nu \epsilon_4 = \frac{2\Delta'_4}{b/2} = \frac{\nu \sigma_4}{E}$$

$$\Delta'_4 = \frac{\nu}{4E} b \sigma_4$$

$$\sum M = 0 \Rightarrow X_4 = \frac{E}{3} \cdot \frac{b}{a} \cdot \Delta_4$$

$$Y_4 = 0$$

Bending in y direction.



$$\epsilon_5 = \frac{2\Delta_5}{b} = \frac{\sigma_5}{E}$$

$$\Delta_5 = \frac{1}{2E} b \cdot \sigma_5$$

$$\nu \epsilon_5 = \frac{2\Delta'_5}{a/2} = \frac{\nu \sigma_5}{E}$$

$$\Delta'_5 = \frac{\nu}{4E} a \cdot \sigma_5$$

$$\sum M = 0 \Rightarrow X_5 = 0$$

$$Y_5 = \frac{E}{3} \cdot \frac{a}{b} \cdot \Delta_5$$

	u_1	v_1	u_2	v_2	u_3	u_4	u_4	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4
Strain in x-direction	0	0	0	0	Δ_1	0	Δ_1	0	$\frac{E}{1-\nu^2} \frac{b}{a} \frac{\Delta_1}{2}$	$\frac{\sqrt{E}}{1-\nu^2} \frac{\Delta_1}{2}$	$\frac{E}{1-\nu^2} \frac{b}{a} \frac{\Delta_1}{2}$	$\frac{\sqrt{E}}{1-\nu^2} \frac{\Delta_1}{2}$	$\frac{E}{1-\nu^2} \frac{b}{a} \frac{\Delta_1}{2}$	$\frac{\sqrt{E}}{1-\nu^2} \frac{\Delta_1}{2}$	$\frac{E}{1-\nu^2} \frac{b}{a} \frac{\Delta_1}{2}$
Strain in y-direction	0	0	0	Δ_2	0	Δ_2	0	$\frac{\sqrt{E}}{1-\nu^2} \frac{\Delta_2}{2}$	$\frac{E}{1-\nu^2} \frac{a}{b} \frac{\Delta_2}{2}$	$\frac{\sqrt{E}}{1-\nu^2} \frac{\Delta_2}{2}$	$\frac{E}{1-\nu^2} \frac{a}{b} \frac{\Delta_2}{2}$	$\frac{\sqrt{E}}{1-\nu^2} \frac{\Delta_2}{2}$	$\frac{E}{1-\nu^2} \frac{a}{b} \frac{\Delta_2}{2}$	$\frac{\sqrt{E}}{1-\nu^2} \frac{\Delta_2}{2}$	$\frac{E}{1-\nu^2} \frac{a}{b} \frac{\Delta_2}{2}$
Shear Strain	$-\Delta_3$	0	Δ_3	0	Δ_3	0	$-\Delta_3$	0	$\frac{E}{1+\nu} \frac{a}{b} \frac{\Delta_3}{2}$	$\frac{E}{1+\nu} \frac{a}{b} \frac{\Delta_3}{2}$	$-\frac{E}{1+\nu} \frac{a}{b} \frac{\Delta_3}{2}$	$-\frac{E}{1+\nu} \frac{a}{b} \frac{\Delta_3}{2}$	$\frac{E}{1+\nu} \frac{a}{b} \frac{\Delta_3}{2}$	$-\frac{E}{1+\nu} \frac{a}{b} \frac{\Delta_3}{2}$	$-\frac{E}{1+\nu} \frac{a}{b} \frac{\Delta_3}{2}$
Bending in x-direction	Δ_4	$-\Delta_4'$	$-\Delta_4$	$-\Delta_4'$	Δ_4	$-\Delta_4'$	$-\Delta_4$	$-\Delta_4'$	0	$\frac{E}{3} \frac{b}{a} \Delta_4$	0	$\frac{E}{3} \frac{b}{a} \Delta_4$	0	$-\frac{E}{3} \frac{b}{a} \Delta_4$	0
Bending in y-direction	$-\Delta_5'$	$+\Delta_5$	$-\Delta_5'$	$-\Delta_5$	$-\Delta_5'$	$-\Delta_5$	$-\Delta_5'$	$-\Delta_5$	0	$-\frac{E}{3} \frac{a}{b} \Delta_5$	0	$-\frac{E}{3} \frac{a}{b} \Delta_5$	0	0	$\frac{E}{3} \frac{a}{b} \Delta_5$
$\Delta_1 = \frac{1}{2}$ $\Delta_3 = \frac{1}{4}$ $\Delta_4 = \frac{1}{4}$ Rigid body translation $-\frac{1}{4}$ in y-direction	0	0	0	0	1	0	0	0	U_{1u3}	V_{1u3}	U_{2u3}	V_{2u3}	U_{3u3}	V_{3u3}	U_{4u3}
$\Delta_2 = \frac{1}{2}$ $\Delta_3 = \frac{1}{4}$ $\Delta_5 = \frac{1}{4}$ Rigid body translation Δ_5 in x-direction	0	0	0	0	0	1	0	0	U_{1u3}	V_{1u3}	U_{2u3}	V_{2u3}	U_{3u3}	V_{3u3}	U_{4u3}
$\Delta_1 = \frac{1}{2}$ $\Delta_3 = -\frac{1}{4}$ $\Delta_4 = -\frac{1}{4}$ Rigid body translation $-\frac{1}{4}$ in y-direction	0	0	0	0	0	1	0	0	U_{1u4}	V_{1u4}	U_{2u4}	V_{2u4}	U_{3u4}	V_{3u4}	U_{4u4}

APPENDIX B.

An extension of the Finite Element method applied to the plate bending problems.

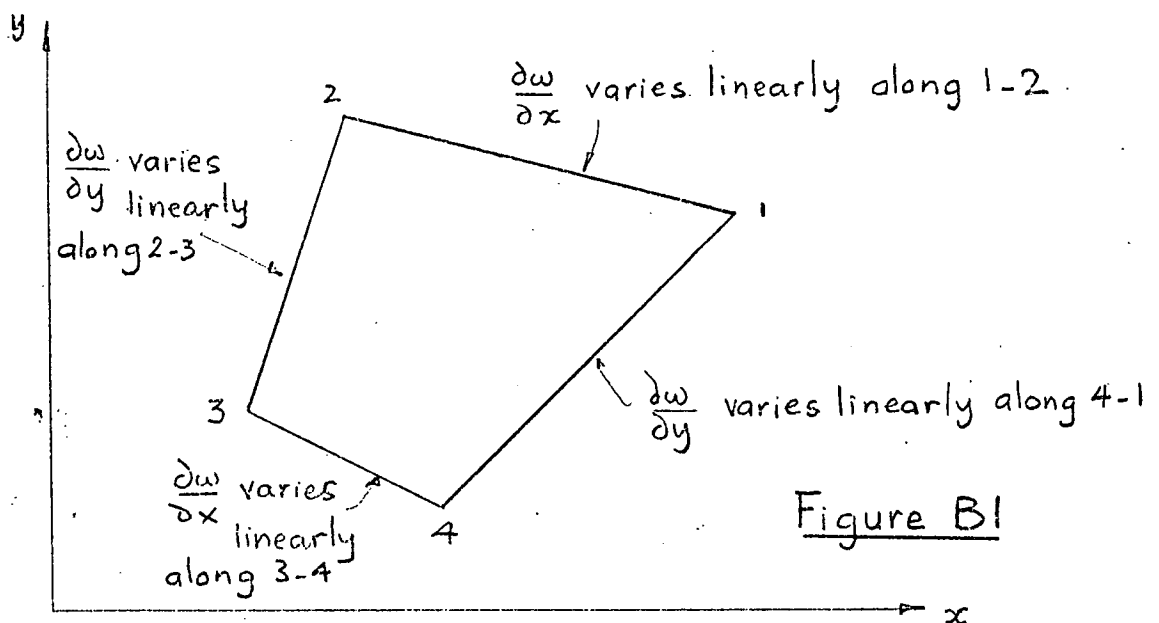
The Finite Element method usually starts with an arbitrary division of the structure into a number of elements. A shape function is chosen to define the state of deformation within an element. As shown in Chapter III, the shape function represents a combination of some basic modes of deformation of the element. Thus the accuracy of the analysis depends upon the versatility of the chosen shape function and the number of elements used. Normally, a relatively simple shape function is used therefore a large number of element is required to achieve satisfactory results.

On the experimental side, the Moire fringe methods give total pictures of deformation of a structure through fringes which have special geometrical meanings. The diffraction grating method gives contours of displacement, the Ligtenberg method gives contours of slope.

If the experimental information about the deformation of a structure is used in the Finite Element analysis, (in place of arbitrary division of elements and arbitrary shape function), some advantages may be realized e.g. fewer elements without any undue loss of accuracy, quicker convergence...

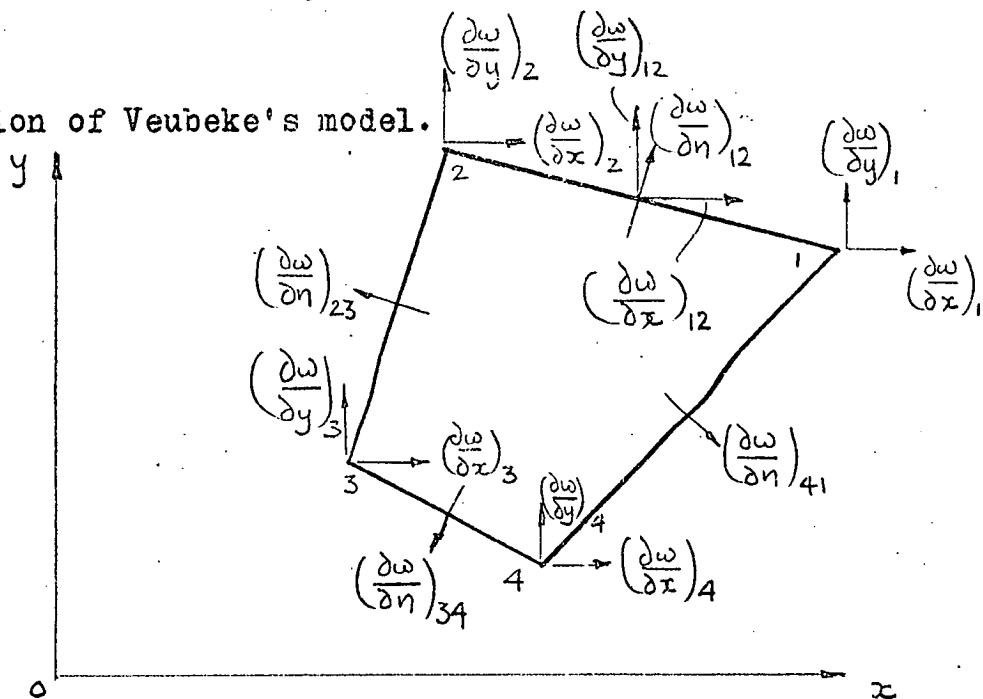
For the plate bending problem, it is shown here that by aligning the elements along lines of linearly varying slope, fewer elements are needed without any loss of accuracy.

In order to do this, one needs a program for arbitrary quadrilateral elements with some special characteristics as shown in Fig. B1.



F.Veubeke's Finite Element model for arbitrary quadrilateral element³ is modified to give the above characteristics.

Modification of Veubeke's model.



The Veubeke's model³ has 16 degrees of freedom. Deflection and two slopes at each corner and the normal slope at the mid-point of the side. With a small transformation, one can replace the later degree of freedom (normal slope at mid-point of the side) into slope in x or y direction at that point.

Along side 1-2 for example, the deflection varies as a cubic. The slopes $\frac{\partial \omega}{\partial x}$, $\frac{\partial \omega}{\partial y}$ vary as a quadratic. The slope along side 1-2 $\frac{\partial \omega}{\partial x}$ can be made to vary linearly by imposing :

$$\left(\frac{\partial \omega}{\partial x}\right)_{12} = \frac{1}{2} \left[\left(\frac{\partial \omega}{\partial x}\right)_1 + \left(\frac{\partial \omega}{\partial x}\right)_2 \right]$$

where $\left(\frac{\partial \omega}{\partial x}\right)_{12}$ = slope at mid point of side 1-2.

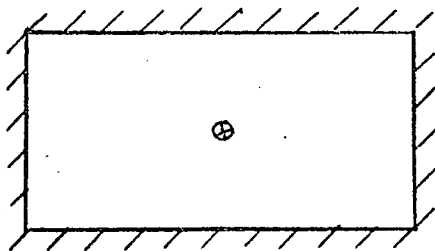
$\left(\frac{\partial \omega}{\partial x}\right)_1$ = slope at node 1 .

$\left(\frac{\partial \omega}{\partial x}\right)_2$ = slope at node 2 .

Similarly, along side 2-3 $\frac{\partial \omega}{\partial y}$ can be made to vary linearly.

The element is thus reduced to one with 12 degrees of freedom.

Analysis of a clamped plate with a central concentrated load.



Stainless Steel model

$$E = 30 \times 10^6 \text{ psi}$$

$$h = 0.023''$$

$$P = 20 \text{ lbs.}$$

- (i) Contours of slopes in x and y directions were obtained with the Moire Ligtenberg method.
- (ii) From the photographs, lines along which the slopes in x or y directions vary linearly are obtained. From these lines a net of elements are chosen so that each element has the properties

described above. (see Fig. B2)

(iii) This set of elements is analysed with the modified program.

Comparision of results.

The results of the above analysis are compared with that of the analytical method (Timoshenko¹) and that of a normal Finite Element program using rectangular elements (Zienkiewicz²).

For the cental deflection, the following figures are obtained.

	N ⁰ of elem.	N ⁰ of nodes	Degrees of freedom	w
Present Program	10	20	60	0.178"
Rectangular elements	20	30	90	0.180"
Analytical method				0.176"

Comparision of deflections and slopes along the center lines of of the plate is presented in Graph B1 .

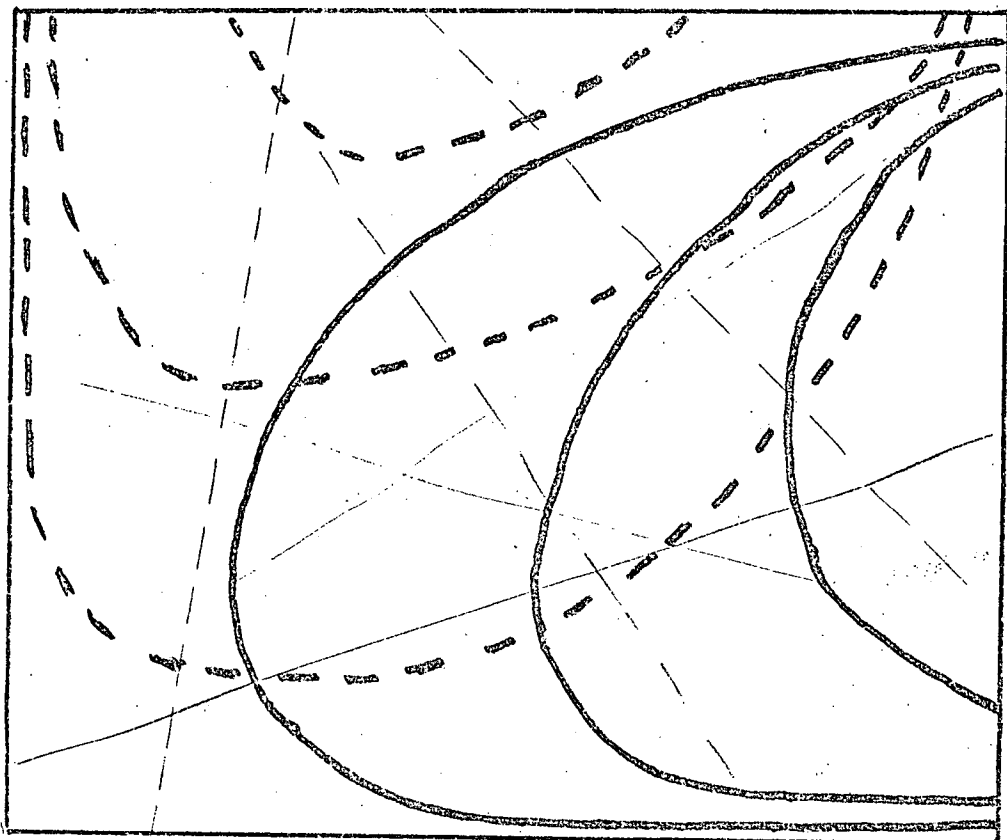
Acknowledgement.

The general idea of this development was suggested by Professor A.R.Oliver to whom the author is greatly indebted.

References.

1. Timoshenko, S.P & Woinowsky-Krieger, S. Theory of plates and shells. Mc-GrawHill(1959).
2. Zienkiewicz, O.C. The Finite Element Method in Structural and Continuum Mechanics. Mc-GrawHill(1968).
3. Fraeijs de Veubeke, B. A conforming finite element for plate bending. Int. Jour. of Solids and Structures. N^o1 Vol.4, 1968.

Figure B2

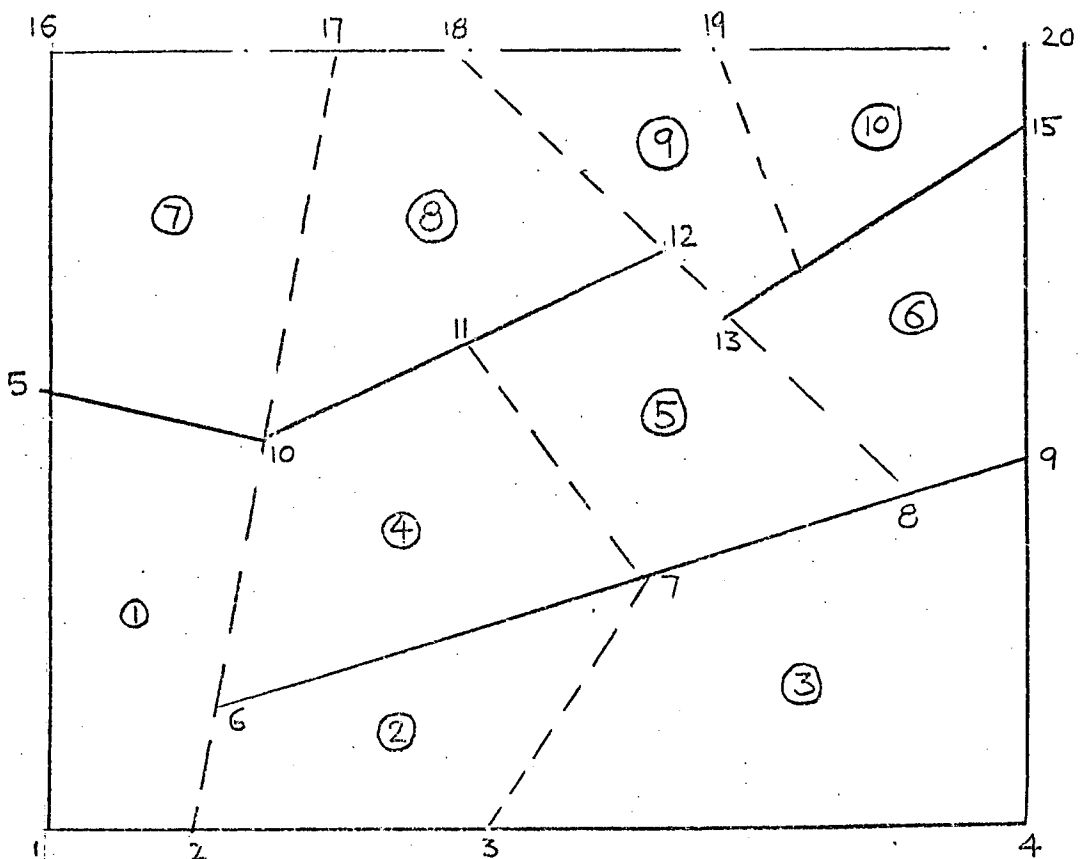


--- Contour of $\frac{\partial w}{\partial x} = \text{constant}$.

— Contour of $\frac{\partial w}{\partial y} = \text{constant}$.

--- Line of linearly varying $\frac{\partial w}{\partial x}$.

— Line of linearly varying $\frac{\partial w}{\partial y}$.



Division of elements

